Projection Filtering in Image Processing

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Abstract

In this paper we shall consider the new projection scheme of local image processing of the visual information. It is based on an expansion into series of eigenfunctions of the Fourier transform. This scheme can be used for compression of images and any kind other media data, their filtration, tracing of outlines, definition of structures and properties of objects.

Keywords: Fourier transform, Hermite functions, image processing.

1. INTRODUCTION

Fourier analysis plays a very important role in image processing, image analysis and, more generally, in signal processing. At the same time, image parameterization to code image information by some kind of mathematical formulae enables to perform many of image processing procedures in a most effective way. The aim of the work is to show an effective possibility to use both approaches simultaneously.

The proposed method is based on the features of Hermite functions. An expansion of signal information into a series of these functions enables one to perform information analysis of the signal and its Fourier transform at the same time, because the Hermite functions are the eigenfunctions of Fourier transform. It is also necessary to underline that the joint localization of Hermite functions in the both spaces makes this method very stable to information errors.

This functions are widely used in pure mathematics, where the expansion into Hermite functions is also called as Gram-Charlier series [1],[2]. They are also used in image processing [3],[4], where the expansion is called as Hermite transform. Nevertheless, these series are usually "limited to the first few terms". The same situation is typical for Hermite function's use in physics, etc. (see some references in [5]).

This work illustrates some possibilities to take full advantage of the use of this method of projection Fourier filtering, mathematically justified in [5].

2. HERMITE FUNCTIONS

The Hermite functions satisfy an important feature for image processing, as they derivate a full orthonormal in $L_2(-\infty, \infty)$ system of functions.

The Hermite functions are defined as:

$$\psi_n(x) = \frac{(-1)^n e^{x^2/2}}{\sqrt{2^n n! \sqrt{\pi}}} \cdot \frac{d^n(e^{-x^2})}{dx^n}$$

They also can be determined by the following recurrent formulae:

$$\psi_0 = \frac{1}{\sqrt[4]{\pi}} \cdot e^{-x^2/2}$$

$$\psi_1 = \frac{\sqrt{2}x}{\sqrt[4]{\pi}} \cdot e^{-x^2/2}$$

$$\psi_n = x\sqrt{\frac{2}{n}} \cdot \psi_{n-1} - \sqrt{\frac{n-1}{n}} \cdot \psi_{n-2}, \forall n \geq 2$$

Moreover the Hermite functions are the eigenfunctions of the Fourier transform:

$$F(\psi_n) = i^n \psi_n,$$

where F denotes Fourier transform operator.

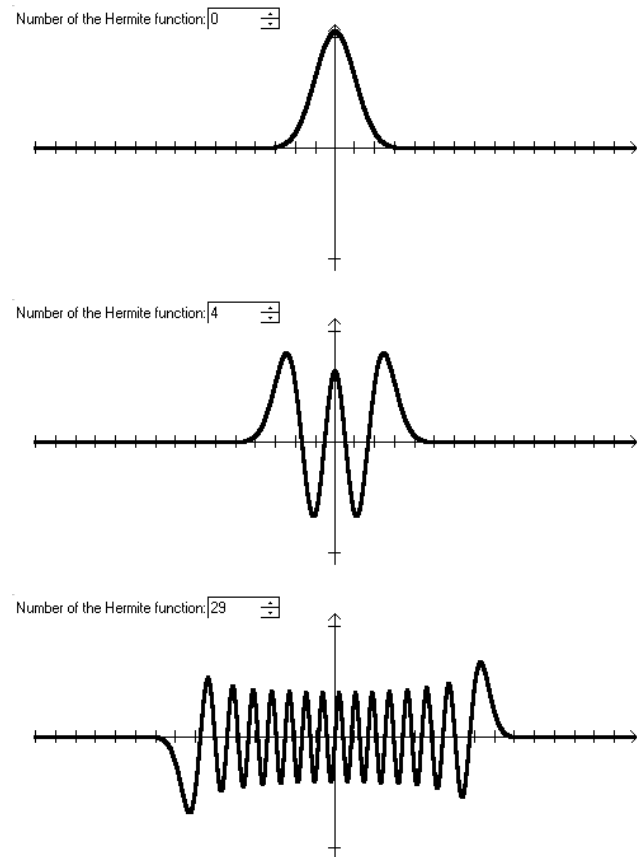The graphs of the Hermite functions look like the following:

Figure 1: Hermite functions

3. THE ALGORITHM

The algorithm represented here works for true color images, but for a simplicity we shall consider only the use of the algorithm for grayscale images, as any true color image can be presented as the aggregate of three grayscale images.

3.1 Base lines

First, we must emit base lines, because

$$\psi_n(x) \to 0, |x| \to \infty$$

So if we have image $I[j,i]$, $i=0..width$, $j=0..height$, than baselines are determined as:

$$baseline_j(i) = I[j,0] + \frac{I[j,width] - I[j,0]}{width} \cdot i$$

Further for every line of an original image (fig. 2) we subtract calculated baseline from the original values and center result concerning value's axis.

Figure 2: Original image

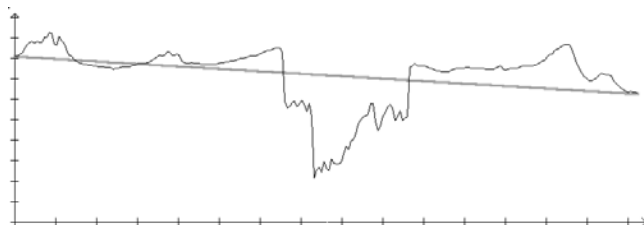**Figure 3:** Baselines

Figure 4: Baseline (wide line)

and original line (tight line) for $j=30$

Now gained image is ready for image processing.

3.2 Approximated lines

At this stage, at first, we should select the number of the Hermite functions used for filtration. Further we stretch our approximation's segment $[-A_0, A_0]$ to the segment $[-A_1, A_1]$, defined from the next criteria:

$$\int_{-A_1}^{A_1} \psi_n^2(x)dx = 0.99,$$

where n is the number of the Hermite functions for the approximation.

Then we decompose function $f(x)$ gained by subtraction of the baseline from j lay of the original image into Fourier series:

$$value(x) = \sum_{i=0}^{n-1} c_i \psi_i(x)$$

$$c_i = \int_{-A_1}^{A_1} f(x)\psi_i(x)dx$$

Since the Hermite functions are the eigenfunctions of Fourier transform, we have also found Fourier transform of the approximation for j lay (fig. 5) of the original image.
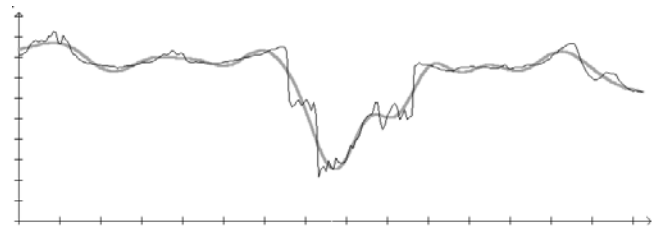

Figure 5: Approximated line (wide line)

and original line (tight line) for $j=30$
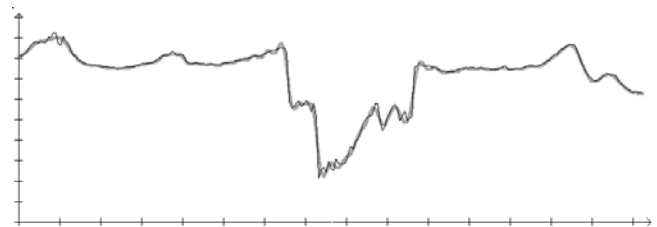
by 20 Herimte functions

Figure 6: Approximated line (wide line)

and original line (tight line) for $j=30$

by 80 Herimte functions

3.3 1D pass

Approximating each line of our image we shall receive 1D filtered image. The number of functions taken here is identical for all levels. Therefore the obtained template of the original image is determined only by baselines and coefficients of expansion for each level.

The result of 1D filtering by this algorithm of the original image (fig.2) is illustrated with figures 7-10.
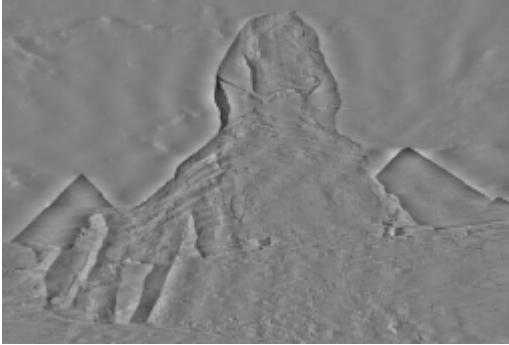
Figure 7: Decoded image by 20 Hermite functions

Figure 8: Difference image by 20 Hermite functions
(+50% intensity)

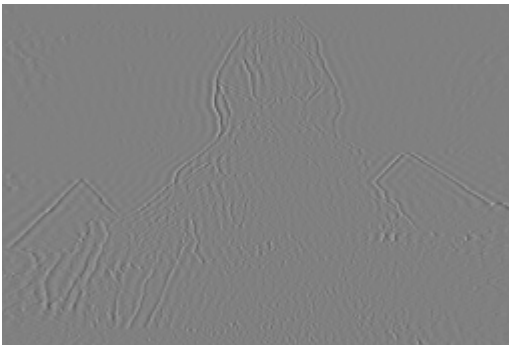**Figure 9:** Decoded image by 80 Hermite functions

Figure 10: Difference image by 80 Hermite functions
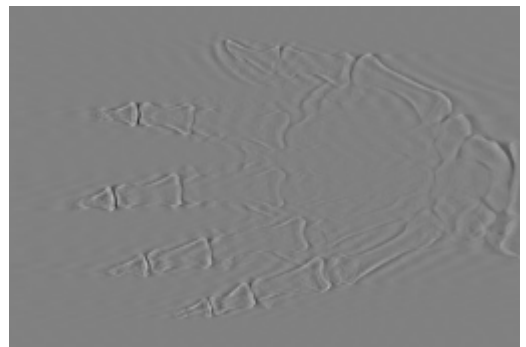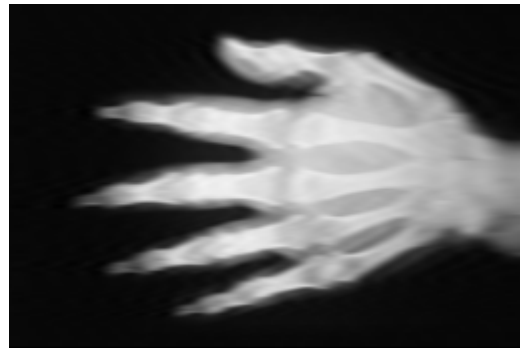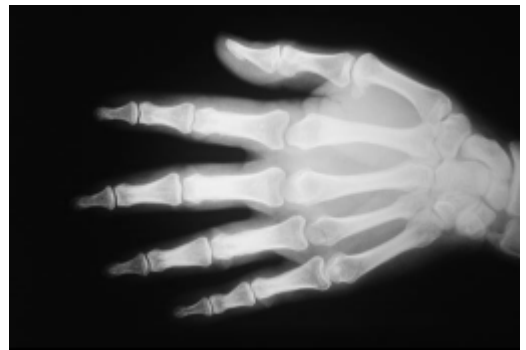(+50% intensity)

Figure 11: Original image,
decoded image by 40 Hermite functions and
difference image by 40 Hermite functions (+50% intensity)

3.4 2D pass

If we consider the gained template of the original image as a new image, rotated by 90°, and do all previous calculations with it, we receive 2D filtered image (fig. 12). The number of functions for this second pass can be different from the number of functions used for the first pass. Therefore the obtained 2D template is determined only by baselines and coefficients of expansion for each level of 1D template.

In case of 1D pass image processing happens line by line. Therefore for tasks immediately connected to tracing of objects, filtering and compressing, it is better to use 2D pass.

Figure 12: 2D decoded image by 80 Hermite functions at the first pass and 60 Hermite functions at the second pass
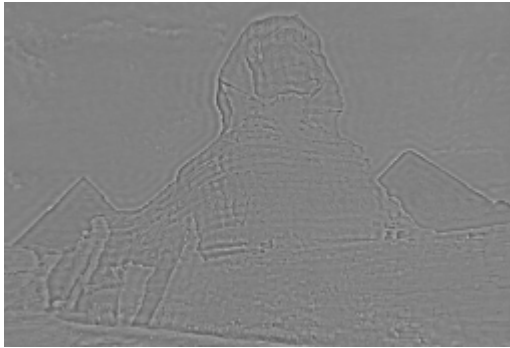

Figure 13: 2D difference image by 80 Hermite functions at the first pass and 60 Hermite functions at the second pass (+50% intensity)

4. CONCLUSION

The Hermite functions are used in this work to filter an image. These functions enable us to separate "decoded image" as a low frequency part of the original image and a high frequency "difference image". Here, the concept of frequency corresponds to the performing Fourier transform operation and is based on the series of Hermite functions. This series is analogous to trigonometric Fourier raw, but Hermite functions are used for the case of an infinite interval while the trigonometric Fourier raw is used for a finite interval.

The approach based on this concept of frequency looks promising to be used in different problems of image and signal processing.

5. REFERENCES

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