Modeling BRDF by a Probability Distribution

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Abstract

In this work we investigate reciprocity and energy conservation properties of a BRDF model which is based on a probability distribution. Representational ability of underlined model is demonstrated using a measured BRDF data set. We show that our model satisfies both reciprocity and energy conservation properties of BRDF.

Keywords: *BRDF model, Rendering, Copula distribution.*

1. INTRODUCTION

A complete representation of surface reflection behavior through a mathematical model can be achieved by considering a number of variables including angle of light, wavelength, polarization and position etc. both for incoming and for outgoing lights. A common approach is to use a simplified model by considering the directional and spectral properties of underlying reflection only. A function of incident and reflected angles is called as the Bidirectional Reflectance Distribution Function (BRDF) defined as

$$\rho(\vec{\omega}_i, \vec{\omega}_o) = \frac{dL_o(\vec{\omega}_o)}{L_i(\vec{\omega}_i)\cos\theta_i d\vec{\omega}_i}, \tag{1}$$

where L_i and L_o are the incident and reflected radiance, respectively, $(\vec{\omega}_i, \vec{\omega}_o) = \{(\theta_i, \phi_i), (\theta_o, \phi_o)\}$ are the corresponding incoming and outgoing vectors, and $d\vec{\omega}_i$ is the differential solid angle in the ω_i direction.

A good BRDF model should obey both reciprocity and energy conserving principles. Generally, it is difficult to build a BRDF model that satisfies these two principles. However, some of the BRDFs have managed to produce visually acceptable images even though they violate energy conservation or reciprocity or both [1].

An interesting approach for modeling the surface reflectance would be to treat the BRDF in the context of probability theory. After a convenient normalization, the BRDF can be viewed as a four-dimensional (4D) probability density function of elevation and azimuth angles of incoming and outgoing vectors. A major problem of this approach is that it is difficult to construct a multivariate probability distribution that provides an adequate approximation for a given BRDF. The most well known multivariate distributions have the same marginal distributions. For example, the marginal distributions of multivariate normal distribution are all univariate normal. Generally, BRDF samples do not exhibit such a property. For example, the elevation angles have extremely skewed distributions but the azimuth angles have U-shaped symmetric distributions for most isotropic materials [2]. A straightforward solution to the problem of modeling the BRDF through probability distributions could be transforming the underlying variables into four independent variables and expressing the BRDF as a product of univariate probability density functions (pdfs) of these new random variables. An algorithm has been proposed by Lawrence at al. [3] who have factorized the 4D BRDF in the directions of incident and outgoing directions and have expressed the BRDF as a conditional distribution for a given outgoing direction vector. Thus, for a fixed outgoing direction, the problem is reduced to a two-dimensional (2D) case. Furthermore, the 2D pdf also is factored to obtain a product of two univariate pdfs. However, this method requires storing a large data set for the factorization procedures. In a recent work, Öztürk et al. [2] proposed to model BRDF by employing copula distributions. Using Archimedean family of distributions for modeling BRDFs, it is shown empirically that the BRDF can be represented successfully by the underlying distribution.

In this work we explain the rationale behind representing the BRDF by copula distributions. We also show that this model satisfies both reciprocity and energy conservation principles.

2. PREVIOUS WORK

Various models have been proposed for approximating the BRDF. These models can be classified in two main groups: analytical models and data-driven models.

The Phong model [4] is one of the oldest and perhaps the most well known analytical BRDF model. Later, using the halfway vector, the Blinn-Phong model [5] has been introduced as an improved version of the Phong model. Ward [6] introduced a BRDF model for describing the isotropic and anisotropic materials. This model is based on Gaussian distribution. Duer [7] proposed a variation of the Ward model. This model has different normalization factor from the Ward model and this improves the fitting results [8]. The BRDF model, developed by Lafortune et al. [9], can be considered as an improved and generalized Phong model. It can represent non-Lambertian diffuse reflections, retro reflections and Fresnel effects. Ashikhmin and Shirley [10] introduced a model for describing anisotropic materials with an improved Phong model. Edwards et al. [1] has modeled the BRDF with a probability distribution. Other empirical models include the models by Lewis [11] and by Westlund and Meyer [12].

While aforementioned analytical BRDF models are phenomenological, there exist physically-based analytical BRDF models. These physically-based analytical BRDF models [13], [14], [15], [16], [17] are based on microfacet theory and can represent effects such as Fresnel reflection and rough microgeometry. Models that can represent anisotropic materials have been developed by Kajiya [18], Poulin and Fournier [19], Ward [6], Lafortune et al. [9], Ashikhmin and Shirley [10], Duer [7] and Edwards et al. [1].

Other class of analytical BRDF models consists of linear combination of some set of basis functions. For describing the BRDF with linear models, Spherical Harmonics were used by Westin et al. [20]. Similarly, Zernike polynomials were used by Koenderink et al. [21] to model the BRDF. Linear models based on Wavelets [22], [23] and PCA [24] have also been used for modeling the BRDF. Ozturk et al. [25] presented an approach based on principal component transformations of some explanatory variables for approximating both isotropic and anisotropic reflectance for diffuse and glossy surfaces. These models require a large number of coefficients to represent the BRDF. Therefore they are inefficient in terms of storage space and computation complexity.

Data-driven models generally depend on large data sets. Matusik et al. [24] acquired dense measurements of the BRDF and used these measurements directly in rendering process. Since this representation requires large storage space, accurate and compact representations of measured BRDFs have been investigated [26], [27], [28], [3], [29], [30]. In these models, BRDF data are decomposed using various factorization methods.

3. COPULA DISTRIBUTIONS

Copula is a multivariate cumulative distribution function of the uniform random variables on the interval $[0, 1]$. They provide a simple and general structure for modeling multivariate distributions through univariate marginal distributions. More specifically if $F_1(x_1)$ and $F_2(x_2)$ are the continuous marginal cumulative distributions then the joint cumulative distribution function of X_1 and X_2 can be constructed in the following form

$$H(x_1, x_2) = C\{F_1(x_1), F_2(x_2)\} = C(u_1, u_2), \tag{2}$$

where C is the copula function, $u_1 = F_1(x_1)$ and $u_2 = F_2(x_2)$ [31]. It is well known that $F_1(x_1)$ and $F_2(x_2)$ are uniformly distributed on the interval $[0, 1]$ and if these cumulative distributions are continuous then the corresponding copula function is unique [31]. Based on this definition, it is clear that the marginal distributions can be determined independently after an appropriate copula model is chosen to represent dependency between the marginal distributions.

An important family of copulas, called Archimedean copulas, has a simple form with some desirable properties. These copula distributions mostly have closed forms and simple solutions. One particularly simple form of a *p*-dimensional Archimedean copula is

$$C(u_1, u_2, \ldots, u_p) = \varphi^{-1}\{\varphi(u_1) + \varphi(u_2) + \ldots + \varphi(u_p)\}, \tag{3}$$

where the function φ is known as *generator* and φ^{-1} is the inverse generating function.

Any function satisfying the following properties

- $\varphi(1) = 0$
- $\lim_{t \to 0} \varphi(t) = \infty$
- $\varphi'(t) < 0$

is defined as a generator. For example, the Frank distribution is obtained from the generating function

$$\varphi(t) = \ln\left\{\frac{\exp(\alpha t) - 1}{\exp(\alpha) - 1}\right\}, \alpha \neq 0, \tag{4}$$

where α is the parameter of the distribution. This generator has the inverse form as

$$\varphi^{-1}(s) = \frac{1}{\alpha}\ln\{1 + \exp(s)(\exp(\alpha) - 1)\}, \tag{5}$$

and the derivation of corresponding copula is straightforward:

$$\begin{aligned} &H(x_1, x_2, \ldots, x_p) = C(u_1, u_2, \ldots, u_p) \\ &= \varphi^{-1}\left\{\ln\left\{\frac{\exp(\alpha u_1) - 1}{\exp(\alpha) - 1}\right\} + \cdots + \ln\left\{\frac{\exp(\alpha u_p) - 1}{\exp(\alpha) - 1}\right\}\right\} \\ &= \frac{1}{\alpha}\ln\left\{1 + \frac{(\exp(\alpha u_1) - 1)\cdots(\exp(\alpha u_p) - 1)}{\exp(\alpha) - 1}\right\}. \end{aligned} \tag{6}$$

Finally the joint pdf of $X_1, X_2, \ldots, X_p$ is obtained by taking the *p*th partial derivative of these variables giving

$$f(x_1, x_2, \ldots, x_p) = \frac{\partial^p}{\partial x_1 \partial x_2 \cdots \partial x_p} C(u_1, u_2, \ldots, u_p). \tag{7}$$

Thus, the pdfs of 2D and three-dimensional (3D) Frank distribution can be found as

$$\begin{aligned} &f(x_1, x_2) = \frac{\alpha g_1(1 + g_{u_1+u_2})}{(g_1 + g_{u_1} g_{u_2})^2} \prod_{i=1}^{2} f_i(x_i), \\ &f(x_1, x_2, x_3) \\ &= \frac{\alpha^2 g_1^2 (1 + g_{u_1+u_2+u_3})(g_1^2 - g_{u_1} g_{u_2} g_{u_3})}{(g_1^2 + g_{u_1} g_{u_2} g_{u_3})^3} \prod_{i=1}^{3} f_i(x_i), \end{aligned} \tag{8}$$

where $g_t = \exp(\alpha t) - 1$ and f_1 , f_2 and f_3 are the marginal pdfs. It is seen from the equations above, the dependency between the variables are defined through the Frank copula with one unknown parameter α.

4. MODELING THE BRDF BY PROBABILITY DISTRIBUTIONS

Suppose for simplicity that the measured BRDF data is obtained by sampling, uniformly along each azimuth and elevation angles of incident and outgoing directions. Let $(\theta_i^r, \phi_i^s, \theta_o^t, \phi_o^u)$ denote a sample point at which the corresponding measured BRDFs $(\widetilde{b}_{rstu})$ are obtained. Up to a constant, we can view that the measured BRDF values are the densities of a continuous joint probability function of $\theta_i, \phi_i, \theta_o$ and ϕ_o. In other words we assume that the measured BRDFs are proportional to the normalized frequencies of corresponding classes (bins). The normalizing coefficient can be determined in such a way that the volume of corresponding joint pdf is equal to 1. When the measured BRDF data is obtained by a uniform spacing with length 1 degree in the intervals $0 \leq \theta_i, \theta_o < 90°$ and $0 \leq \phi_i, \phi_o < 180°$ then the normalized BRDF values can be obtained as

$$b_{rstu} = \frac{\widetilde{b}_{rstu}}{b_{....}}, \tag{9}$$

where $r, t = 0, 1, \ldots, 89$; $s, t = 0, 1, \ldots, 179$ and

$$b_{....} = \sum_{r=0}^{89} \sum_{s=0}^{179} \sum_{t=0}^{89} \sum_{u=0}^{179} \widetilde{b}_{rstu}. \tag{10}$$

Based on this definition, the empirical marginal distributions can be determined easily. For example, the empirical marginal pdf for θ_i is obtained by summing over the other three variables as

$$f_{\theta_i}^r = b_{r...} = \sum_{s=0}^{179} \sum_{t=0}^{89} \sum_{u=0}^{179} b_{rstu}, \tag{11}$$

where $r = 0, 1, \ldots, 89$. Similarly, the empirical joint pdf of θ_i and ϕ_i is determined as

$$f_{\theta_i, \phi_i}^{r,s} = b_{rs..}, \tag{12}$$

where $r = 0, 1, \ldots, 89$ and $s = 0, 1, \ldots, 179$. Marginal pdfs of $\theta_i, \phi_i, \theta_o$ and ϕ_o are shown in Figure 1 for an isotropic material. It is seen that the marginal pdfs of θ_i and θ_o both are skewed to the right and they are almost the same. This similarity property also is observed for the U-shaped marginal pdfs of ϕ_i and ϕ_o. The joint pdf of θ_i and ϕ_i is illustrated in Figure 2 for the same material.

Modeling BRDFs by probability distributions using the standard coordinate system whose coordinates are defined by $\theta_i, \phi_i, \theta_o$ and

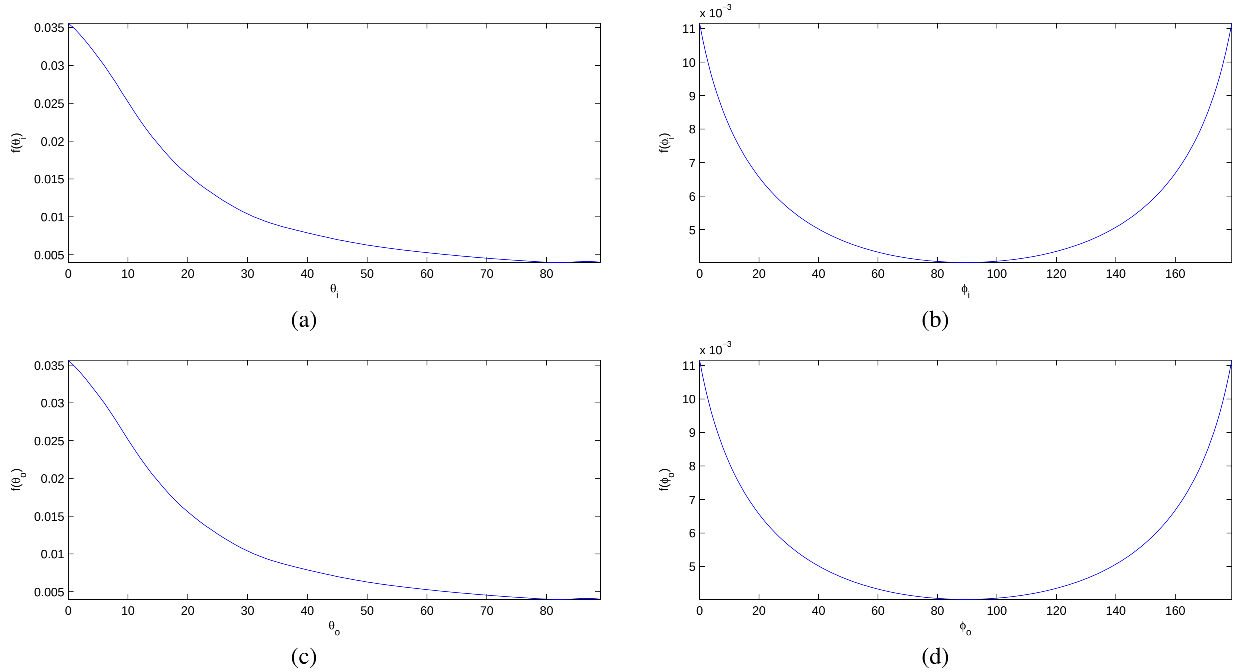

Figure 1: Empirical marginal distributions of elevation and azimuth angles for incident and outgoing directions (blue-metallic-paint).

ϕ_o provide a considerable flexibility. For a given outgoing direction, the distribution of $\vec{\omega}_i = (\theta_i, \phi_i)$ can be any bivariate pdf. However, such a representation usually does not produce a visually plausible BRDF [1]. Based on the measured BRDF data, Ngan et al. [8] have shown that the halfway vector representation of BRDF yields more visually plausible result than that of the standard representation. In this work we proceed to use the halfway vector representation of Rusinkiewicz [32] for isotropic materials. This representation depends on the halfway vector $\vec{\omega}_h = (\theta_h, \phi_h)$ and the difference vector $\vec{\omega}_d = (\theta_d, \phi_d)$. It is well known that isotropic BRDFs are independent of the angle ϕ_h and thus the corresponding pdf can be expressed as a function of three variables namely θ_h, θ_d and ϕ_d.

For an isotropic material the corresponding 3D BRDF copula model can be written as

$$\rho(\theta_h, \theta_d, \phi_d) = Kc(u_1, u_2, u_3; \alpha) f_1(\theta_h) f_2(\theta_d) f_3(\phi_d), \quad (13)$$

where $u_1 = F_1(\theta_h), u_2 = F_2(\theta_d), u_3 = F_3(\phi_d)$, c is 3D copula pdf, α is the copula parameter and K is the scaling coefficient obtained in a similar way for this 3D case as in Equation (10). The distribution functions F_1, F_2 and F_3 are the cumulative marginal distribution functions of θ_h, θ_d and ϕ_d, respectively. These marginal distributions are estimated directly from the measured BRDF data.

5. ESTIMATION

Since we treated the normalized BRDF measurements as the observed probability densities corresponding to a bin in a 4D histogram, we can assume that the number of observations (frequencies) falling into these bins for this 3D case can be expressed in a similar way as

$$n_{rst} = n b_{rst}, \quad (14)$$

where $n = n_{\dots}$ represents the total number of frequencies which is unknown for our case. Any well known statistical estimation

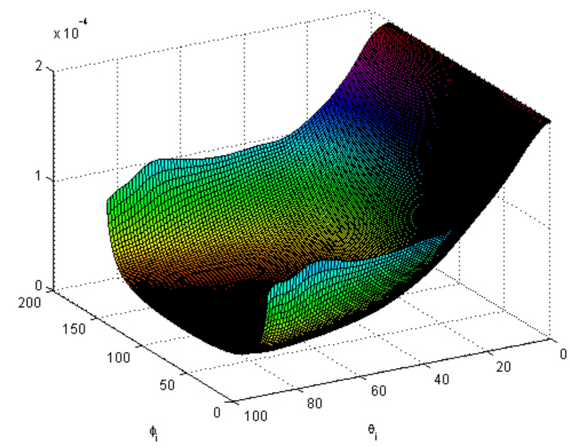

Figure 2: The joint pdf of θ_i and ϕ_i for blue-metallic-paint.

method can be adopted for estimating the unknown parameters of hypothesized distribution. For example, the maximum likelihood technique can be used straightforwardly for this purpose.

Based on the data matrix whose rows are organized as follows

$$\{b_{rst}, \theta_h^r, \theta_d^s, \phi_d^t\}, \quad (15)$$

the corresponding log-likelihood function for our special case can be written as

$$l(\lambda) = n \sum_r \sum_s \sum_t b_{rst} \log\{f(\lambda; \theta_h^r, \theta_d^s, \phi_d^t)\}, \quad (16)$$

where f is the joint pdf and λ is the vector of unknown parameters. The Archimedean copula distribution used in this work has a single parameter α. It is interesting to note that the maximization of above objective function with respect to the parameters does not depend on the unknown sample size n. Thus the estimates of parameter can be obtained as a function of normalized measured BRDFs.

One difficulty in modeling BRDF by a multivariate probability distribution is the number of unknown parameters which is prohibitively large for a practical application. This problem could be overcome if the corresponding probability model can be constructed as a function of marginal distributions of underlying variables. In such case the estimation process can be performed in two stages: first the parameters of marginal distributions can be determined separately and then they are substituted in the objective function to estimate remaining parameters [33]. In this sense the copula distributions provide a unified approach both for modeling and estimating the BRDF.

Considering some practical difficulties that are encountered in the tails of distributions, we employed the least squares technique instead of the maximum likelihood technique for estimating the dependency parameter of copula pdf. The least squares estimation procedure is explained in detail in [2]. The objective function for obtaining the non-linear least squares estimate of α is defined as

$$S(\alpha) = \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{r}\{b_{ijk} - c(F_1^{(i)}, F_2^{(j)}, F_3^{(k)}; \alpha) f_1^{(i)} f_2^{(j)} f_3^{(k)}\}^2, \tag{17}$$

where $f_1^{(i)} = b_{i..}, f_2^{(j)} = b_{.j.}, f_3^{(k)} = b_{..k}$ and F_1, F_2 and F_3 are cumulative marginal distributions.

We have observed that the marginal distributions of θ_h for specular materials are extremely skewed [2]. Our empirical results showed that copula distributions do not provide satisfactory approximations for these cases. We overcome this difficulty by dividing data along θ_d into subsamples and fitting the BRDF model to each of these subsamples. In this work, we obtained 6 subsamples by dividing θ_d into 6 non overlapping intervals each with a length of $90^\circ/6 = 15^\circ$.

6. PROPERTIES OF THE MODEL

A physically plausible BRDF model should obey the reciprocity and the energy conservation properties. Our BRDF model lends itself to satisfy both reciprocity and energy conservation properties.

Reciprocity: The reciprocity property of BRDF is expressed by the following equation

$$\rho(\vec{\omega}_i, \vec{\omega}_o) = \rho(\vec{\omega}_o, \vec{\omega}_i). \tag{18}$$

That is the BRDF is unchanged when the order of $\vec{\omega}_i$ and $\vec{\omega}_o$ is changed. Our model depends on the halfway vector representation. To enforce the reciprocity condition we use the identity

$$\phi_d = \phi_d + \pi. \tag{19}$$

Under this enforcement, our model satisfies the reciprocity principle.

Energy conservation: Energy conservation property of a BRDF model dictates that for every outgoing light direction

$$\int_\Omega \rho(\vec{\omega}_i, \vec{\omega}_o)(\vec{\omega}_i \cdot \mathbf{n}) d\vec{\omega}_i \leq 1. \tag{20}$$

Our BRDF model depends on a multivariate probability distribution function but it is scaled with a coefficient K. In this sense, our model may not be considered as an energy conserving model. However, one can modify this coefficient to account for absorption of different wavelengths of light as suggested by Edwards et al. [1].

7. IMPORTANCE SAMPLING

In this section we describe a sampling technique that can be used for our BRDF representation. Our main goal is to evaluate the following integral for a given outgoing direction:

$$L_o(\vec{\omega}_o) = \int_\Omega L_i(\vec{\omega}_i)\rho(\vec{\omega}_i, \vec{\omega}_o)\cos\theta_i d\vec{\omega}_i, \tag{21}$$

where Ω stands for unit hemisphere above the surface. We need to develop an efficient sampling procedure since direct or numerical evaluation of the underlying integral is difficult. A common approach to tackle this problem is to use variance reduction techniques. This technique exploits the fact that an estimator $(1/n)\sum_{i=1}^{n} h(X_i)/f(X_i)$ where f is the pdf of X and n is the sample size, is an unbiased estimator of the integral $\int h(x)dx$ and if the function f is similar to the function h then this statistic converges to the integral more quickly.

At a given pixel position and an outgoing direction, the incident illumination integral is given by Equation (21) where $\rho(\vec{\omega}_i, \vec{\omega}_o)$ is the BRDF based on our representation. Since we assume that the outgoing direction is known, we can use the conditional pdf of $\vec{\omega}_i$ given that $\vec{\omega}_o$ to estimate the integral value as

$$L_o(\theta_o, \phi_o) = \frac{1}{n}\sum_{i=1}^{n} L_i(\theta_i, \phi_i)\frac{\rho(\theta_i, \phi_i, \theta_o, \phi_o)\cos\theta_i \sin\theta_i}{f(\theta_i, \phi_i | \theta_o, \phi_o)}, \tag{22}$$

where the conditional pdf in the denominator can be expressed in terms of the BRDF and the joint marginal density function h of θ_o and ϕ_o as

$$f(\theta_i, \phi_i | \theta_o, \phi_o) = \frac{\rho(\theta_i, \phi_i, \theta_o, \phi_o)}{h(\theta_o, \phi_o)}. \tag{23}$$

The marginal distribution of θ_o and ϕ_o can be estimated from the measured BRDF data. Substituting this conditional pdf in Equation (22), we get the following simplified expression

$$L_o(\theta_o, \phi_o) = \frac{1}{n} h(\theta_o, \phi_o) \sum_{i=1}^{n} L_i(\theta_i, \phi_i)\cos\theta_i \sin\theta_i. \tag{24}$$

It is interesting to note that the resulting Monte Carlo estimator of the illumination integral does not contain the BRDF and it can be computed easily for each incident vector. However, the corresponding incident vector has to be generated from the conditional distribution of θ_i and ϕ_i in Equation (23). It can be shown that the joint cumulative conditional distribution function of θ_i and ϕ_i is [34]

$$F(\theta_i, \phi_i | \theta_o, \phi_o) = \frac{\varphi^{-(2)}\{\varphi(u_1) + \varphi(u_2) + \varphi(u_3) + \varphi(u_4)\}}{\varphi^{-(2)}\{\varphi(u_1) + \varphi(u_2)\}}, \tag{25}$$

where $u_1 = F_1(\theta_o), u_2 = F_2(\phi_o), u_3 = F_3(\theta_i), u_4 = F_4(\phi_i)$ and $\varphi^{-(2)}$ stands for the second derivative of the inverse generating function. An algorithm for generating random numbers from Frank distribution is provided by Genest and Rivest. However, generating random variates from the conditional bivariate Frank distribution using Equation (25) is not straightforward. For this purpose we generate two standard uniform random variates ξ_1 and ξ_2 first and then generate θ_i as the solution of

$$\xi_1 = F_{\widetilde{\theta}_i}(\widetilde{\theta}_i | \theta_o, \phi_o), \tag{26}$$

and ϕ_i as the solution of

$$\xi_2 = F_{\widetilde{\phi}_i}(\widetilde{\phi}_i | \theta_o, \phi_o, \widetilde{\theta}_i). \tag{27}$$

These nonlinear equations can be solved using numerical techniques. Our work is being continued along the line of finding approximate closed form solutions for these equations.

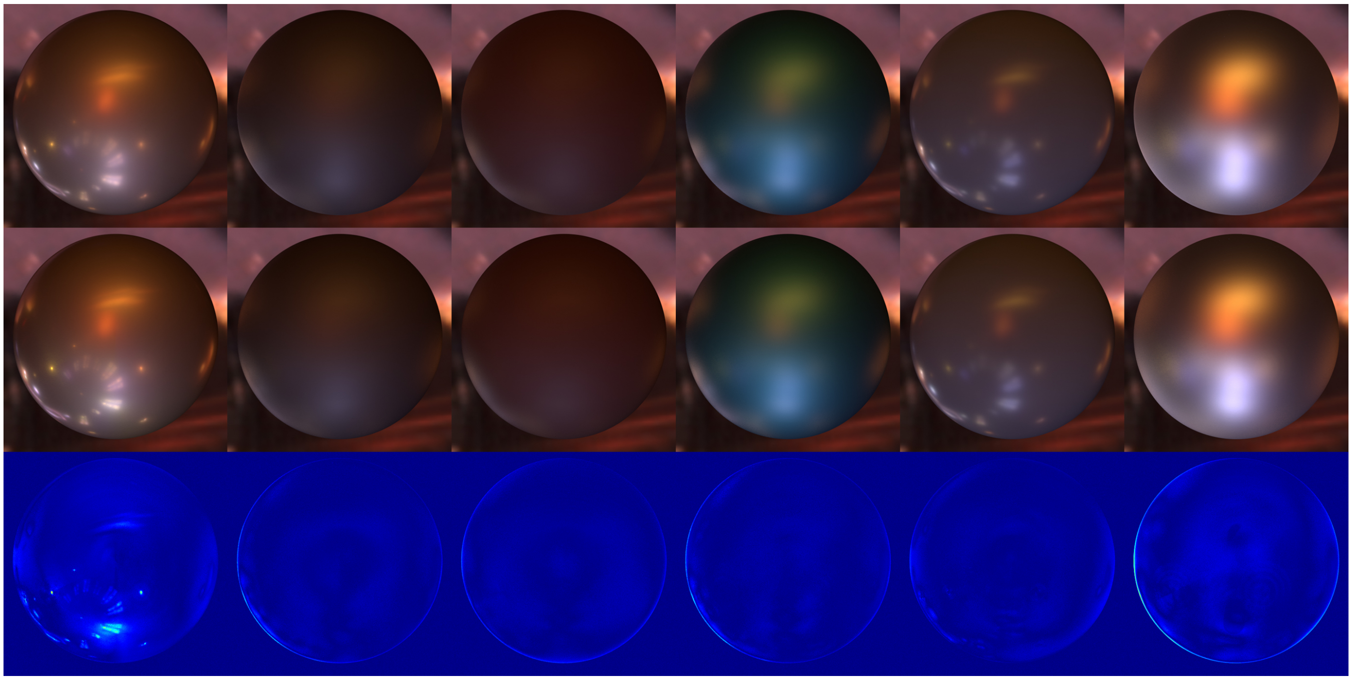

Figure 3: Various spheres were rendered with our Frank copula model using different materials. Columns left to right: alum-bronze, black-oxidized-steel, dark-specular-fabric, green-metallic-paint, pvc and silver-metallic-paint. Rows top to bottom: Reference images were rendered using measured data; images were rendered using our Frank copula model and color-coded difference images (Color-coded differences are scaled by a factor of five to improve the visibility of differences between the real and approximated images).

8. RESULTS

In this work we used the measured BRDF data of Matusik et al. [24] for isotropic materials. This data is sampled using the halfway representation because this representation is more suitable both for specular and for diffuse materials. All materials in this data set are sampled uniformly with a resolution $90 \times 90 \times 180$ for $(\theta_h, \theta_d, \phi_d)$, respectively. We have obtained empirical marginal distributions of θ_h, θ_d and ϕ_d and estimated the unknown parameters of Archimedean copula pdfs using this data set. To illustrate the quality of approximation of proposed model we fitted 3D Frank copula model to 6 different materials which include diffuse, glossy and specular materials. The results are presented in Figure 3. The spheres shown on the last row of figure are color-coded difference images [35] to improve the visibility of differences between the real and approximated images. Based on these materials the Frank copula model has produced satisfactory approximations for the real images.

Our model satisfies the two important properties of BRDF namely the reciprocity and energy conservation. The reciprocity property of our model is illustrated in Figure 4. In this figure we interchanged the incoming and outgoing direction vectors and rendered the Princeton scene [3], [1]. It is clearly seen in the color-coded difference image that the reciprocity property of our BRDF model is satisfied for this special case.

The energy conservation property of the model is demonstrated by using the procedure of Neumann et al. [36]. We evaluated the integral of Equation (20) for each outgoing directions and for three different isotropic materials. The results are shown in Figure 5. In these calculations no modification is made on the coefficient K. It is seen that our model satisfies the energy conserving property for each of the three materials. This property of our model has been verified by considering various isotropic materials.

9. CONCLUSION

In this paper we considered the problem of representing BRDF through a multivariate probability distribution and discussed some related problems. Archimedean family of copula distributions can be used to model the BRDF as a function of the corresponding cumulative marginal distributions. This class of distributions not only provides a simple functional form for modeling the BRDF but also lends itself to satisfy the reciprocity and energy conservation properties of an ideal physically-based BRDF.

Modeling the BRDF by a probability distribution has another advantage that is the Monte Carlo estimator of the illumination integral can be simplified and hence can be evaluated without using the estimated BRDF function. In this case, however, the incoming vector should be sampled using an appropriate simulation technique.

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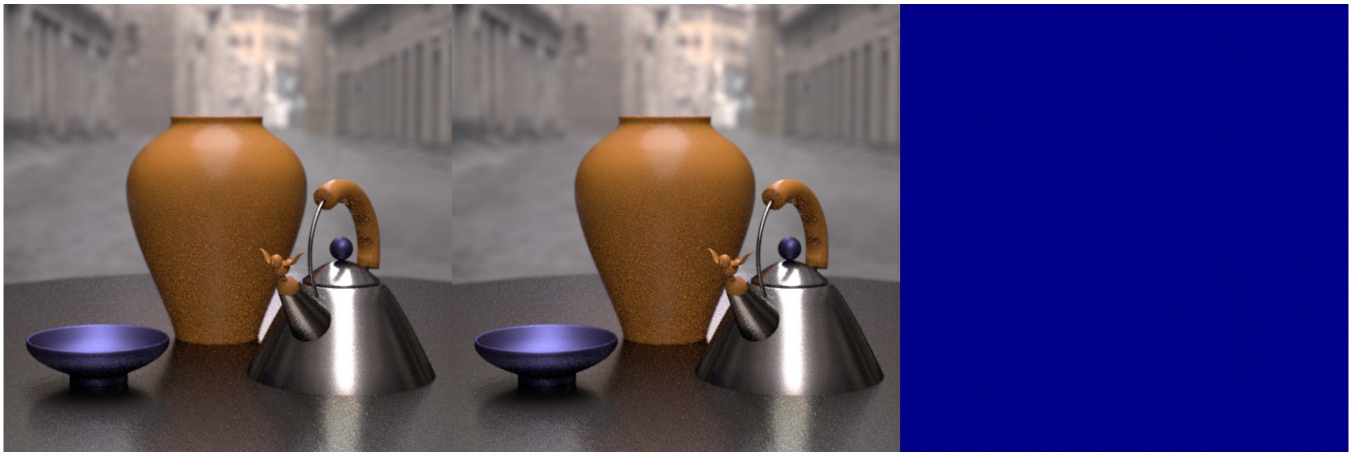

Figure 4: Left scene was rendered using Frank copula model. Middle scene was rendered using the same Frank copula model with incoming and outgoing vectors interchanged. Right scene shows a color-coded difference between left and middle images.

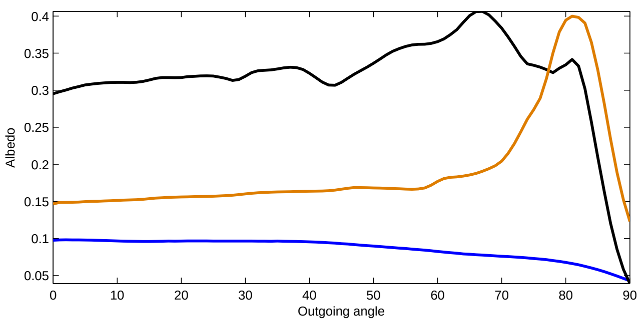

Figure 5: Albedo functions of Frank copula model for three measured materials: blue-metallic-paint (blue), yellow-matte-plastic (orange) and nickel (black).

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