Post-processing by Total Variation Quasi-solution Method for Image Interpolation

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Abstract

Image restoration is one of classical inverse problems in image processing and computer vision, which consists of the recovering information about the original image from incomplete or degraded data. In this paper, we consider the problem of reduction of ringing that appears after image resampling. We introduce a novel method for image restoration, based on a quasi-solution method for a compact set of functions with bounded total variation. It is an alternative approach to using a total variation functional as a stabilizer in Tikhonov regularization, and it does not oversmooth or displace edges.

Keywords: *total variation, image interpolation, deringing.*

1. INTRODUCTION

Many problems of image restoration can be posed as problems of solution of a linear equation

$$Az = u, \qquad z \in Z,\ u \in U, \tag{1}$$

where Z and U are Hilbert spaces, A is a linear continuous operator from Z to U, so the inverse operator A^{-1} exists but is unbounded. Thus, the problem (1) is ill-posed [1, 2] and the corresponding matrix for operator A is ill-conditioned.

One of the ways to make the problem (1) well-conditioned is using *Tikhonov regularization method* [1, 2]. It makes this problem well-posed and prevents noise amplification during restoration. This method constructs the approximation $\tilde{z}$ of the unknown source function $\bar{z}$ from the observed degraded (noisy) function $u_\delta : \|A\bar{z} - u_\delta\| \le \delta$.

$$\tilde{z} = \arg\min_{z \in Z} \int_\Omega [(Az - u_\delta)^2 + \alpha \Psi(|\nabla z|^2)]dx, \tag{2}$$

where the regularization parameter $\alpha = \alpha(\delta)$ is chosen in accordance with the noise level. In image processing, following classes of functions Ψ are typically used: (a) Tikhonov functional $\Psi(t) = t$, (b) total variation $\Psi(t) = \sqrt{t}$. The utilization of Tikhonov functional leads to a quadratic problem, but strongly smoothes sharp edges. Total variation method allows finding discontinuous solutions, so it preserves edges during restoration.

We have discussed this method for image interpolation in our previous paper "Image Interpolation by Super-Resolution" [3] and formulated it in discrete form as a problem of minimization of (2) with a more complicated stabilizer:

$$\tilde{z} = \arg\min_{z \in Z} \left(\|Az - u_\delta\|_1 + \alpha \sum_{s,t=-p}^{s,t=p} \gamma^{|s|+|t|} \left\| z - S_x^s S_y^t z \right\|_1 \right), \tag{3}$$

where S_x^s and S_y^t are shift operators along x and y axes by s and t pixels respectively, $\gamma = 0.8$ and A is a downsampling operator. We approximated the solution of (3) by iterative steepest descent method

$$\begin{aligned} x_{n+1} = x_n - \beta_n \{ & A^T sign(Ax - y) + \\ & + \alpha \sum_{s,t=-p}^{s,t=p} \lambda^{|s|+|t|} (I - S_x^{-s} S_y^{-t}) sign(x - S_x^s S_y^t x) \} \end{aligned} \tag{4}$$

More details about it can be obtained in [3].

Despite of good results, Tikhonov regularization method has one disadvantage: it is needed to specify the regularization parameter α, but we cannot define an accurate rule for its selection. The alternative approach to (2) is the minimization problem

$$\begin{gathered} \tilde{z} = \arg\min_{z \in M} \int_\Omega |Az - u_\delta|^2 dx, \text{ where} \\ M = \{ z \in Z \mid \int_\Omega \Psi(|\nabla z|^2) dx \le C \} \end{gathered} \tag{5}$$

In our work, we are using $\Psi(t) = \sqrt{t}$, so M is a set of functions with limited total variation. The problem (5) can't be used directly in image resampling due to several limitations on operator A, but may be very useful for post-processing because parameter C can be reasonably specified.

The rest of the paper is organized as follows. In section 2, we introduce a quasi-solution method for bounded total variation functions for solving problem (1), which does not need to specify noise level or regularization amount. In section 3, we show several applications of this method. Some improvements of this method are described in section 4. Section 5 concludes the paper by summarizing applications of this method in image processing.

2. QUASI-SOLUTION METHOD

Definition. Point $z_K \in M$ for which $\|Az - u\|$ reaches a minimum on a given compact set M of the space Z is called *quasi-solution* [4,5] on M for a given u

$$z_K = \arg \inf_{z \in M} \|Az - u\| \tag{6}$$

If we assume that operator A is continuous, the discrepancy $\|Az - u\|$ will be continuous functional, which reaches its infimum on a compact set M. Thus, a quasi-solution exists for every $u \in U$. The problem of finding quasi-solutions is well-posed [2].

2.1 Quasi-solution method for bounded total variation functions (TVQ)

We applied quasi-solution method (6) for solving problem (1) in one-dimensional case ($Z = L_2[a,b], U = L_2[c,d]$). The set M of constrained functions with variation less than constant $C \ge 0$ is a compact set in $L_2[a,b]$ space, and operator A is linear, bijective and precisely defined. So we consider the following total variation quasi-solution method (TVQ) to solve problem (1) in one-dimensional case: we construct a sequence that minimizes the discrepancy functional $F(z) = \|Az - u\|^2$ on the set of bounded functions with total variation less than given value C.

It is necessary to underline that TVQ method does not need information on the noise level δ in contrast to Tikhonov regularization method. Instead of regularization parameter we use the value of signal total variation as the stabilizing parameter.

2.2 Numerical scheme

For the first time a numerical method for solving TVQ problem has been considered in the book [6].

The discrepancy functional $F(z) = \|Az - u\|^2$ is a quadratic function that is defined for every z on compact set M. This functional is convex and differentiable, and its Frechet derivative is equal to

$$F'(z) = 2(A^* Az - A^* u),$$

here $A^* : U \to Z$ is the adjont operator.

So for approximate solving of the equation (1), we need to construct a minimizing sequence for the convex and differentiable functional on a closed limited set in Hilbert space.

After discretization, we obtain the following problem: to construct a sequence of vectors $z_l \in R^n$ that minimizes quadratic function $\varphi(z)$ on a convex set V_C , where V_C is a set of vectors $z \in R^n$, which components satisfy conditions:

$$\begin{gathered} |z_2 - z_1| + |z_3 - z_2| + \ldots + |z_n - z_{n-1}| \le C \\ z_n = 0\,. \end{gathered} \tag{7}$$

As $V_a^b(z) = V_a^b(z + c)$, it is natural to fix the value of function z on a boundary of segment $[a,b]$. Thus we assume that we know one of the boundary values $\bar{z}(a)$ or $\bar{z}(b)$ (hereinafter we assume that $\bar{z}(b) = 0$, $z_n = 0$).

Since considered functional has a Frechet derivative satisfying Lipschitz condition with a constant $L = 2\|A\|^2$, the conditional gradient method can be used to solve this problem. It is described in detail in [6]. We have also analyzed an algorithm which doesn't need to fix $z_n = 0$.

3. APPLICATIONS

The proposed TVQ regularization method is applicable for a wide range of signal processing problems:

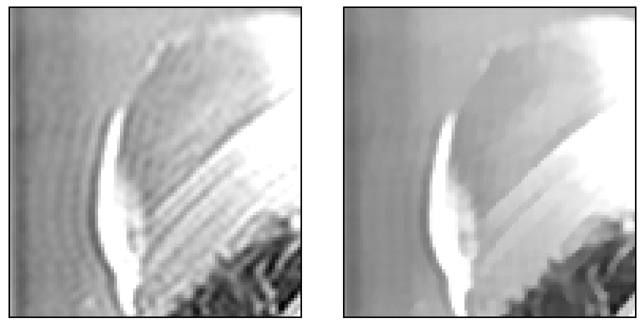
1. Image restoration.
2. Gibbs phenomenon reduction (deringing).
3. Noise reduction (denoising).
4. Super-resolution and interpolation.

Below we will consider application of TVQ method to the problem of deringing after interpolation. Therefore, we assume in (6) $A = I$ (unit operator).

3.1 Gibbs effect reduction (deringing)

Gibbs phenomenon (ringing effect) is caused by quantization or truncation of high frequency information by approximation method. For example, it can be seen after shrinkage of coefficients of Fourier or wavelet transform. In spatial domain, this effect produces spurious oscillations near sharp edges.

TVQ method eliminates ringing effects and practically does not blur edges. For image processing (see a result in figure 1) we are using the algorithm introduced in the previous section



a) Detail of a source image

b) Detail of a smoothed image (decimation coefficient is equal to 0.75)

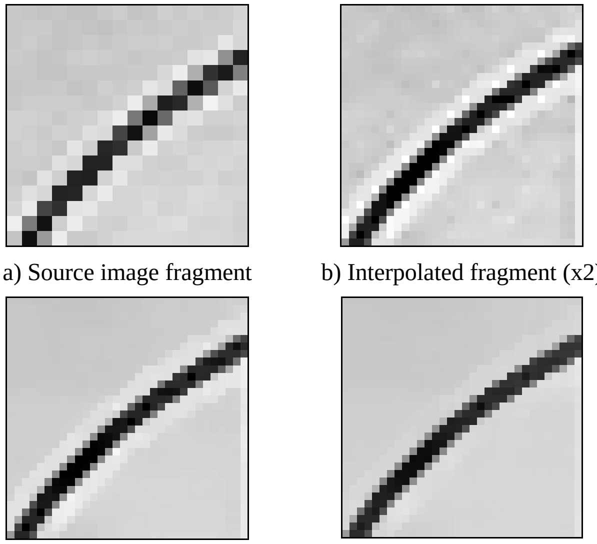
Figure 1: Gibbs phenomenon reduction.

3.2 Interpolation

Many resampling algorithms introduce ringing artifacts, so it's reasonable to apply our method to process interpolated images. Some methods need to define regularization parameter α , but do not provide a way to find its optimal value. Smaller values produce ringing artifacts, higher — smooth image too much. So, we choose small regularization and then use TVQ method to remove ringing artifacts. The key assumption is that the total variation of interpolated image should be the same as for the source image, so we can reasonably define a constant C in (5). In one-dimensional case, the task is formulated in the following way:

1. Source discrete image x with total variation $C = V(x)$ and an interpolated image y are given. For example, image $\tilde{z}$ may be obtained using method (3).
2. We need to construct an image z^* such that: $z^* = \min_{z \in V_{C'}} \|z - \tilde{z}\|$, where $V_{C'} = [z : V(z) \le C']$, $C' = kC$, where k is a coefficient close to 1.

In two-dimensional case, we may divide the image into a set of rows and columns, process them separately and then average results of vertical and horizontal processing. The example result is shown in figure 2. The initial interpolated image was obtained by superresolution method based on a regularization formula (3).

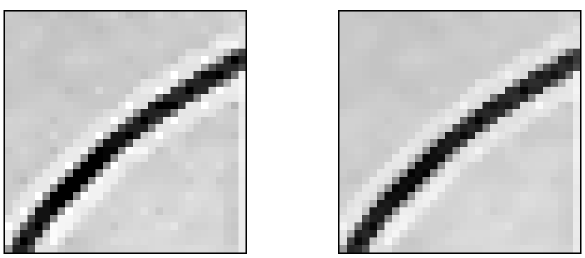


a) Source image fragment | b) Interpolated fragment (x2)

c) Postprocessed (k=1.5) | d) Postprocessed (k=1.2)

Figure 2: Image resampling.

One of drawbacks is the fact that this method operates with entire rows and columns, but Gibbs phenomenon appears only near edges, so we can lose details outside edges. So, we have proposed an adaptive algorithm to avoid this problem: for every pixel in interpolated image we take a small fixed-size square fragment with a center at this pixel and a corresponding fragment on a source image, consider these fragments as new source and interpolated images, and then we process this pair by the algorithm described above. Pixel values of resulting image are constructed from central pixel values of postprocessed fragments.



a) Interpolated fragment | b) Postprocessed by local method (k=1.0)

Figure 3: Image resampling.

Ringing effect was not fully removed because it was present in the source image but this method has removed the effect added by the resampling algorithm. In comparison with the general method, this method works faster because the iteration process converges much faster when operates with the reduced set of points.

4. METHOD IMPROVEMENTS

This method very effectively suppresses ringing artifacts, but it also corrupts high-detailed fragments of an image, for example, grass texture. So, it'll natural to process only areas with perceptible ringing effect near strong edges and skip areas, where no ringing effect or small details are present.

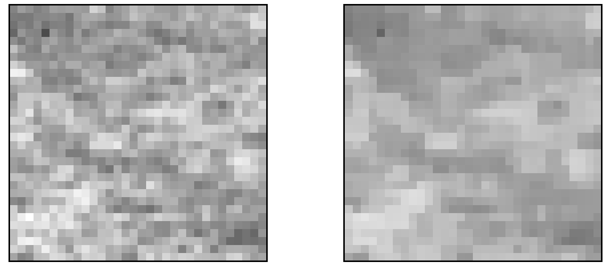


Figure 4: Example of degradation of fine-detailed area before and after processing by TVQ method.

4.1 Weight map

We'll use a weight map to define areas, where ringing effect should be suppressed. The resulting image is constructed as a weighted sum between interpolated image $\tilde{z}$ and interpolated image, postprocessed by TVQ z^*.

$$z_{i,j} = z^*_{i,j} w_{i,j} + \tilde{z}_{i,j}(1 - w_{i,j}), \quad 0 \le w_{i,j} \le 1, \tag{10}$$

The key principle of constructing the weight map is that the weight value is high only near sharp isolated edges.

The weight map is constructed by using the following algorithm:

1. In every point, calculate a norm of gradient which produces edge power map $e_{i,j}$. Low values belong to smooth areas, and high values belong to edges. Higher values result in sharper edges.
2. In every point, calculate the edge score.

$$s_{i,j} = \sum_{r=\sqrt{(i-i')^2+(j-j')^2} \le R} \sigma(f(r)e_{i,j} - p - e_{i',j'}), \quad \text{where } \sigma(x) = \begin{cases} 0, & x \le 0 \\ 1, & x > 0 \end{cases} \tag{11}$$

In other words, $s_{i,j}$ is a number of pixels in a window satisfying a set of conditions

$$f(r)e_{i,j} - p > e_{i',j'} \text{ for each } i', j' : r = \sqrt{(i-i')^2 + (j-j')^2} \le R \tag{12}$$

p is a noise level (the minimum value of edge power to classify a point as a part of an edge), R is a maximum distance between edges with similar power to classify an area as fine-detailed and $f(r)$ is a weight function. We are using the following function

$$f(r) = \frac{1}{2} e^{-\frac{r^2}{2\rho^2}}, \text{ where } \rho = \frac{2}{3} R \tag{13}$$

A one-dimensional example is shown on a figure 5.

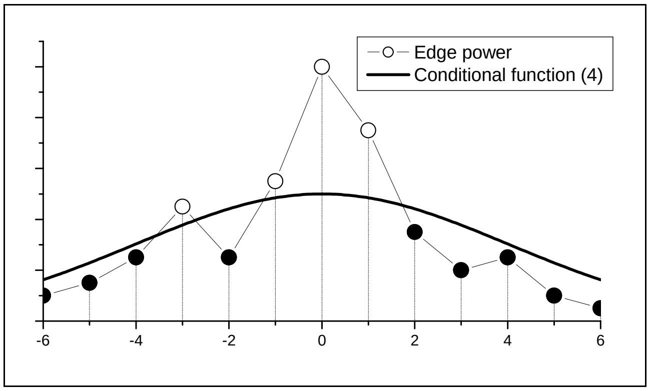


Figure 5: An illustration of the algorithm for R=6. Points, where the condition (12) is true, are solid black. The score is 9.

3. Apply the threshold

$$w_{i,j} = \begin{cases} 0, & s_{i,j} < S \\ 1, & s_{i,j} \ge S \end{cases} \quad (14)$$

S is a minimum number of compliances to classify a point as a part of sharp isolated edge. Sometimes it may be represented as $S = qT$, where T is total number of pixels in a window and q is the percent of edge pixels below the threshold. We assume $q = 0.3$.

4. Filter, dilate and smooth the weight map. Filtering removes isolated single points from the weight map, which usually belong to noise. Dilation is needed to extend areas near edges, because the 3-rd step of this algorithm marks only edges.

4.2 Applications

The use of weight map for image-adaptive filtering is shown in figure 6.

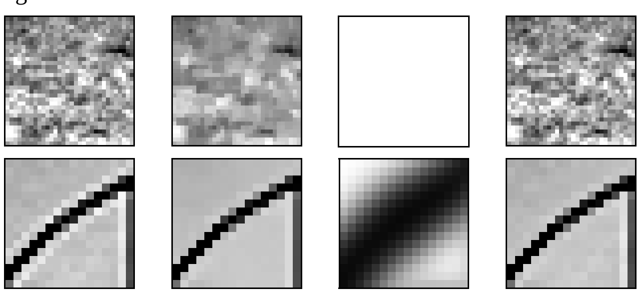


Figure 6: Fragments of source image, processed image, corresponding weight map (inverted) and final image.

We have compared postprocessed interpolated low-resolution images with high-resolution images by PSNR metric and come to a conclusion that use of the weight map increases the PSNR value.

5. CONCLUSION

In this paper, we have suggested a novel image restoration algorithm based on a quasi-solution method for a compact set of functions with bounded total variation. It is an alternative to total variation functional used as the stabilizer in Tikhonov regularization and it also does not oversmooth or displace edges. At the same time, the application of this method does not need estimates of the noise level, which are necessary to choose regularization parameter in the Tikhonov functional. This information on the level of noise is usually unavailable and the selected regularization parameter does not have a reasonable explanation. In our case, we use the information on image total variation value. The approbation of this method with test images shows effectiveness of this method for image deringing and resampling. This quasi-solution method also looks promising for other areas of image processing that traditionally use a total variation approach.

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A program with implemented TVQ method with weight map may be downloaded from the site http://audio.rightmark.org/lukin/graphics/tvq.htm

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