Optimal polygonization of implicit surfaces

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Abstract

In this paper the main definitions of piece regular implicit surfaces, of surfaces curvature, of approximation of piece regular implicit surfaces are introduced. The task of optimal polygonization of piece regular implicit surfaces and some approaches to its solution are given.

Keywords: *computer graphics, 3D modeling, implicit function, implicit surface, regular surface, polygonal surface, curvature, surface curvature, regularization, approximation, triangulation, polygonization, optimal polygonization, mesh optimization.*

1. INTRODUCTION

The task of optimal polygonization of implicit surfaces with features is extensively discussed [2-6]. In this work the class of piecewise regular implicit surfaces is defined and the approaches to a solution of the task of construction optimal polygonization of surfaces from this class are described.

2. POLYGONAL SURFACE

The oriented 2D surface without the boundary, agglutinated from polygons (triangles and tetragons) homomorphic to a circle is called polygonal surface in R^3. Interior area of a polygon of such surface is named a face of the surface, and an edge and a vertices are named an edges and a vertices of the surface. In the further let us assume, that two various faces of polygonal surfaces are not intersected. Thus polygonal surface M in R^3 is the boundary of some body $T(M)$.

In the further, let us assume, that normals to the edges of the surface M defining orientation are directed outside of a body $T(M)$.

3. REGULAR IMPLICIT SURFACE

Let $f : R^3 \to R$ is the smooth image. The set with level c for image f is named regular implicit surface, if it is nonempty and gradient of function f is not becoming zero on this set.

4. CURVATURE

4.1 Curvature of smooth curve

Let $L:[0,d] \to R^3$ is the smooth curve on $(0,d)$, that parameterized length of an arc; $e(s)$ is unit tangent vector to the curve L in the point $L(s)$, where $s \in (0,d)$. Draw $e(s)$ from the center of the unit orb S^2, and then the curve $e:[0,d] \to S^2$ is taken on sphere S^2.

Value of rotation $\sigma_L[a,b]$ of the arc $L[a,b]$ in space is defined as the length of the curve $e:[a,b] \to S^2$ and curvature $k(s)$ of the curve L in the point $L(s)$ is defined as limit:

$$k(s) = \lim_{\Delta \to 0} \frac{\sigma_L[s, s+\Delta]}{\Delta}$$

If length of the curve $e:(0,d) \to S^2$ is finite, the curve L is named as a curve with limited rotation on space. If such value $k_0 > 0$ that for all $s \in (0,d)$ $k(s) \le k_0$ is found, then the curve L is named as curve with limited curvature.

4.2 The rotation of polyline in space

Let L is polyline in R^3 and $A_0, A_1, ..., A_n$ are consecutive vertices of the polyline. Let $e_1, e_2, ..., e_n$ are unit directive vectors of the links $A_0A_1, A_1A_2, ..., A_{n-1}A_n$ of the polyline. Draw the vectors $e_1, e_2, ..., e_n$ from the center of the unit orb S^2. The length of the polyline on the unit orb with consecutive vertices $e_1, e_2, ..., e_n$ is named the rotation of polyline L in space.

If A_s is the interior vertex of the polyline L then any of unit vectors corresponding to point of arc $e_s e_{s+1}$ will be named the tangential vector to L in point A_s.

4.3 The curvature of smooth surface

Let F is the regular implicit surface and let F_P is the tangential space in a point $p \in F$ and e is a unit vector lying in the tangential space F_P and outgoing from a point p, $n(p)$ is a unit normal of a surface in a point p. Set a plane $J(e)$ passing through a point p and containing vectors n and e. This plane intersects a surface along a curve L_e, let $k(e)$ is curvature of a curve L_e in a point p and N is vector outgoing from a point p in direction of center adjoining circle.

The curvature $K(e)$ of the surface in point p in direction e is defined by formula $K(e) = \pm k(e)$ where sign before $k(e)$ is defined as inverse to sign of dot product (N, n).

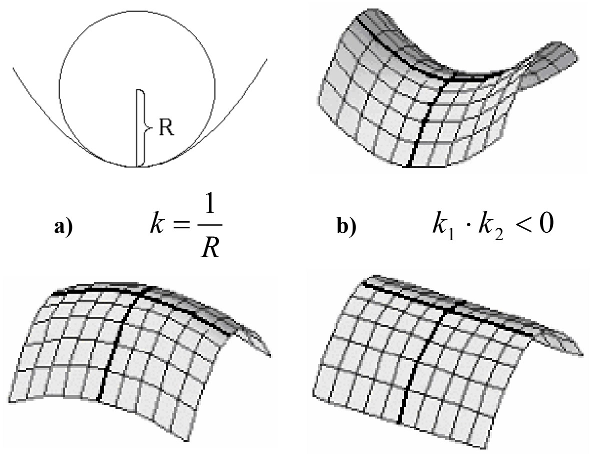
Following theorem is valid:

In tangent space of the point p it is possible to select two orthogonal directions e_1, e_2, such that for any direction $e_\varphi = e_1 \cos\varphi + e_2 \sin\varphi$ value of curvature equally: $k(e_\varphi) = k_1(e_1)\cos^2\varphi + k_2(e_2)\sin^2\varphi$.

Values $k_1(e_1), k_2(e_2)$ is named principal curvatures of a surface in a point p and e_1, e_2 are principal directions appropriate to them.

The value $k = k_1 k_2$ is named Gaussian curvature of a surface in a point p.

Depending on value of Gaussian curvature (fig. 1) the points are called as points of convexity, saddle points, points of zero curvature and point of flattening $(k_1 = 0, k_2 = 0)$.



a) $k = \frac{1}{R}$ b) $k_1 \cdot k_2 < 0$

c) $k_1 \cdot k_2 > 0$ d) $k_1 \cdot k_2 = 0$

Figure 1

Integral curvature of a surface F is the integral

$$\omega(F) = \iint_F K d\sigma$$

where K is the Gaussian curvature and $d\sigma$ the element of surface area.

The set $n(G) = \{n(p) \in S^2 \mid p \in G\}$ on unit orb is named as spherical image of the set $G \subset F$. For the point $p \in F$ the Gaussian curvature is equal to limit of the quotient of areas

$$K(p) = \lim_{G \to p} \frac{m(n(G))}{m(G)}$$

if the areas $G \subset F$ is contracted to the point p.

The integral

$$|\omega|(G) = \iint_G |K| d\sigma$$

is named as absolute curvature of the open set $G \subset F$. [1]

4.4 Calculation of principal curvatures and principal directions

Let P is a point on the surface, e_1, e_2 is a pair of single tangents orthogonal vectors to a surface in the point P, $e_1 = (e_{11}, e_{12}, e_{13}), e_2 = (e_{21}, e_{22}, e_{23})$.

Find the values

$$H_{ab} = -\frac{1}{\|\nabla f\|} \sum_{i,j} \frac{\partial^2 f}{\partial x_i \partial x_j} e_{a,j} e_{b,i}, \quad a, b = 1,2$$

Let k_1, k_2 are the eigenvalues of the matrix H, and $E_1 = (u_{11}, u_{12}), E_2 = (u_{21}, u_{22})$ are the appropriate to them single eigenvector. Then k_1, k_2 are principal curvatures, and $E_1 = u_{11} e_1 + u_{12} e_2, E_2 = u_{21} e_1 + u_{22} e_2$ are principal directions.

4.5 Line of curvature

Curve on a surface F is named as a line of curvature if in each of the own interior points has a tangent which direction coincides with one of principal directions of a surface in the given point.

4.6 The curvature of polygonal surface

Let v is the vertex of polygonal surface M. Let φ is the value of Euclidean angle around of vertex v. This is summation of angles of triangles adjoined to the given vertex.

Then the value $\omega(v) = 2\pi - \varphi$ is named the interior curvature of vertex v.

Let $S(v, \varepsilon) = \{x \in M \mid \rho(x, v) = \varepsilon\}$ is the circle on M centered in vertex v and the circle does not contain inside vertices differing from v. We orient it so that the vertex v remained at the left at bypass of the circle, if view directed by normal (outside of the body $T(M)$). Let S^2 is a unit orb oriented by own exterior normal. Spherical representation $S^*(v, \varepsilon)$ of circle $S(v, \varepsilon)$ is the polyline on the sphere S^2 (on half-sphere), and the consecutive vertices of one are the normals of the face consecutive tracing along $S(v, \varepsilon)$. The oriented area $\overline{\omega}(v)$ of this polyline is named exterior curvature of vertex v. It is easily determined that $\overline{\omega}(v) = \omega(v)$.

Integral curvature of a surface M is the summation $\omega(M) = \sum_v \omega(v)$ where summation is produced along the all vertices of polygonal surface. It is known, that the curvature $\omega(M)$ depend on topological type of surface and is associated with Eiler characteristic of the surface by the following formula $\omega(M) = 2\pi \cdot \chi(M)$. Thus, the curvature of the surface homeomorphic to sphere is equal 4π, the curvature of the surface homeomorphic to torus (to sphere with handle) is equal 0, the curvature of the surface homeomorphic to sphere with k handle is equal $2\pi \cdot (1 - k)$.

The vertex of a polygon surface is called as the regular if the spherical representation of a surface of the oriented circle is a polyline on without self-intersections

The vertex v of a polygon surface M is called as the regular if the spherical representation of a surface $S^*(v,\varepsilon)$ of the oriented circle $S(v,\varepsilon)$ is a polyline on S^2 without self-intersections.

If curvature of such vertex is nonnegative, the vertex is named convex; if curvature of such vertex is negative the vertex is named saddle.

5. PIECE REGULAR IMPLICIT SURFACES

The set P is named as surface in R^3 if it is possessed of the following property: for every point $p \in P$ the such $\varepsilon > 0$ is existed, that the intersection of open sphere $B(p,\varepsilon)$ centered in point p, radius ε with P is homeomorphic to the plain circle B^2. If for some $\varepsilon > 0$ the set $U(p,\varepsilon) = B(p,\varepsilon) \cap P$ allows the regular parameterization $r : B^2 -> R^3$ of class C^3, then point $p \in P$ is named as regular point of surface, else the point p is named as critical point of the surface.

Surface P is named as piecewise-smooth surface, if the set of its critical point Ω is representable as union of two sets Ω_0 and Ω_1, where Ω_0 is the finite set of point and Ω_1 is the finite set of open smooth noncrossing curve with limited curvature; and moreover the absolute curvature of the set $P \setminus \Omega$ on P is limited.

Piecewise-smooth surface is named as piece regular surface if the following condition is additionally held. Let the curve $\gamma \in \Omega_1$ and point $p \in \gamma$. Consider the neighborhood $U(p,\varepsilon)$ which is not contained of critical point except the point from curve γ moreover that $U(p,\varepsilon) \setminus \gamma$ consist of open regular smooth surface U_1, U_2. Require, that each of those surfaces might be continued through a boundary curve $U(p,\varepsilon) \cap \gamma$ by a smooth regular fashion. Thus closed sets $\overline{U}_1, \overline{U}_2$ should be subsets of some open smooth regular surfaces. In the future a points from Ω_0 is named as vertices and curves from Ω_1 is named as edges of piece regular surface.

6. POLYGONIZATION SUBORDINATED TO THE CURVATURE OF THE SURFACE

6.1 Approximation of smooth curve

Let K is smooth curve in R^3, L is polyline. Let's speak, that polyline determines $\varepsilon - \delta$-approximation K, if polyline L has following properties:

1- Polyline L lies in ε-neighborhood of curve K considering as set in R^3.

2 Curve K lies in ε-neighborhood of polyline L considering as set in R^3

3 For every point $p \in L$ the set $\pi_L(p) \subset K$ is defined and for every point $q \in K$ the set $\pi_K(q) \subset L$ is defined and the following conditions are held:

- $\pi_L(p) \subset B(p,\varepsilon)$
- $\pi_K(q) \subset B(q,\varepsilon)$
- $q \in \pi_L \circ \pi_K(q)$
- $p \in \pi_K \circ \pi_L(p)$
- If $t_L(p)$ is a unit tangent to L in point $p \in L$, then such point $q \in \pi_L(p)$ that $|t_K(q) - t_L(p)| < \delta$ is found.
- If $t_K(q)$ is a unit tangent to K in point $q \in K$, then such point $p \in \pi_K(q)$, that $|t_K(q) - t_L(p)| < \delta$ is found.

6.2 Approximation of piece regular implicit surface

Let F is piece regular implicit surface.

The polygonal surface M is named as $\varepsilon - \delta$-approximation F if the following conditions are held:

1 Surface M lies in ε-neighborhood of surface F considering as set in R^3

2 Surface F lies in ε-neighborhood of surface M considering as set in R^3

3 For every point $p \in M$ the set $\pi_M(p) \subset F$ is defined and for every point $q \in F$ the set $\pi_F(q) \subset M$ is defined and the following conditions are held:

- $\pi_M(p) \subset B(p,\varepsilon)$
- $\pi_F(q) \subset B(q,\varepsilon)$
- $q \in \pi_M \circ \pi_F(q)$
- $p \in \pi_F \circ \pi_M(p)$
- If $n_M(p)$ is a unit tangent to M in point $p \in M$, then such point $q \in \pi_M(p)$ that $|n_F(q) - n_M(p)| < \delta$ is found.
- If $n_F(q)$ is a unit tangent to F in point $q \in F$ then such point $p \in \pi_F(q)$ that $|n_F(q) - n_M(p)| < \delta$ is found.

4 Let v is the vertex of polygonal surface M, $U_M(v)$ is the neighborhood, consisting of triangles adjacent to v and $\pi(U_M(v)) \subset D \subset F$.

- If D is the area of strict positive Gaussian curvature, then v is the convex point.
- If D is the area of strict negative Gaussian curvature, then v is saddle point.

The property 4 guarantee to preserve of important characteristics (convexity and saddle-shaped) of corresponding areas on F and M.

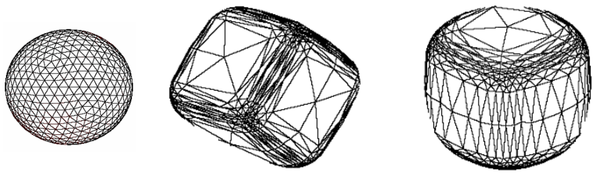
5 Require, that each of edges of the surface F should be accordingly approximated by polyline consisting of edges of polygonal surface M.

6.3 The epsilon-normalized triangulation

Let M is the linear polygonization of surfaces $F = P(f, c)$, Δ is a face of polygonization, $\Delta^* = \pi(\Delta)$ is curvilinear triangle on F, in which Δ is projected on F along lines of a gradient f. Let $n(X)$ is exterior normal to the surface F in a point $X \in \Delta^*$, n is exterior normal to Δ. If an angle $\angle(n, n(X)) \le \varepsilon$ for all $X \in \Delta^*$, then the face Δ is called as ε-normalized. The linear polygonization M of the surface F, which all faces are ε-normalized, and vertices lay on a surface F, is named ε-normalized triangulation of a surface. For the fixed $\varepsilon > 0$, ε-normalized triangulation containing the least amount of triangles is called ε-optimal.

Theorem 1. If a surface F is convex, then under sufficiently small $\varepsilon > 0$ number of triangles ε-optimal triangulation is close to $\dfrac{16\pi}{3\sqrt{3}\varepsilon^2}$.

Theorem 2. Let F is piecewise regular surface, thus $F \setminus \Omega^0$ is disintegrated on connected components $F_1, F_2, ..., F_m$, which are areas of a constant signs Gaussian curvature. Suggest, that the minimum of principal curvatures for points of areas is different from zero and the sum of space rotational of their boundaries is restricted. Then number of triangles ε-optimal triangulation of a surface F under $\varepsilon \to 0$ is equivalent $\dfrac{4K}{3\sqrt{3}\varepsilon^2}$, where *K* is the sum of absolute values of integral curvatures of areas $F_1, F_2, ..., F_m$.



unit orb $\quad x^4 + y^4 + z^4 - 16 = 0 \quad x^4 + y^4 + z^4 + x^2y^2 - 20 = 0$

epsilon-normalized triangulation

Figure 2

7. REFERENCES

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