SECOND ORDER TIKHONOV REGULARIZATION METHOD FOR IMAGE FILTERING

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Abstract

Second order Tikhonov regularization method for image filtering has been designed. Analytical solution of the corresponding functional minimization problem for one-dimensional case was found. Comparisons of the edge detection results for the first and second order Tikhonov regularization image filtering methods was done. It was found that the second order method is more efficient for image segmentation tasks.

Keywords: *image filtering, Tikhonov regularization.*

1. INTRODUCTION

Variational methods are widely used in image filtering. Generally, in order to estimate unknown image *u* from noisy observed one u_δ, we minimize a functional of the form:

$$E_\alpha(u) = \int_\Omega [(u-u_\delta)^2 + \alpha\cdot\varphi(|\nabla u|^2)]dx,$$

where Ω is an open bounded set in R^2 and α is a regularization parameter. Survey of utilized functions φ can be found in Teboul et all [1]. Also this paper analyzes edge-preserving properties of variational methods.

In this paper we suggest to use for image filtering regularization functional using second order derivative. Second order derivatives zero-crossings determination is a basic method to detect step edges. The most popular Marr and Hildreth [2] and Canny edge detector [3] utilize zero-crossings of Laplacian and second order derivative. Thus, the smoothness of the second derivative obtained by our regularization method looks very promising to give good edge-preserving properties and to be used in segmentation tasks.

2. SECOND ORDER TIKHONOV FILTERING METHOD

One-dimensional method based on second order Tikhonov's regularization [4,5] looks as follows:

For an unknown function $\bar{u} \in C_2$, we have an approximation $u_\delta : \|\bar{u} - u_\delta\|_{L_2} \le \delta$. To filter u_δ we find function u_α that minimize the Tikhonov functional:

$$E_\alpha(u_\alpha) = \|u_\alpha - u_\delta\|^2_{L_2} + \alpha \left\| \frac{d^2}{dx^2} u_\alpha \right\|^2_{L_2}.$$

The first term in the functional is a data fidelity term, while the second term rewards smoothness.

The problem of the Tikhonov functional minimization is reduced to solving the Euler equation for u_α:

$$\begin{cases} \alpha u_\alpha^{(4)} + u_\alpha = u_\delta \\ u'_\alpha(1) = u'_\alpha(-1) = 0 \\ u'''_\alpha(1) = u'''_\alpha(-1) = 0 \end{cases},$$

It is significant to use third order derivatives in the boundary conditions to have a self-conjugated problem.

We rewrite this equation as:

$$u_\alpha^{(4)} + 4\lambda^4 u_\alpha = \tilde{u}_\delta,$$

where $\lambda = (4\alpha)^{-1/4}$, $\tilde{u}_\delta = \frac{1}{\alpha} u_\delta$.

Solution of this equation with the given boundary conditions can be found as:

$$u_\alpha(x) = \int_{-1}^{1} G_\lambda(x,s)\tilde{u}_\delta(s)ds,$$

with the kernel that can be obtained by the scheme, described by Nejmark [6]

$$G_\lambda(x,s) = \begin{cases} \sum_{i=1}^{4} a_i(s)u_i(x) & x \le s \\ \sum_{i=1}^{4} b_i(s)u_i(x) & x \ge s \end{cases}.$$

Here u_i are the functions that form fundamental system:

$$u_1(x) = e^{\lambda(x+1)} \sin\lambda(x+1)$$
$$u_2(x) = e^{\lambda(x+1)} \cos\lambda(x+1)$$
$$u_3(x) = e^{-\lambda(x-1)} \sin\lambda(x-1)$$
$$u_4(x) = e^{-\lambda(x-1)} \cos\lambda(x-1)$$

and the coefficients $b_i(s)$ are

$$b_1(s) = -\frac{e^{4\lambda}[p\cos 4\lambda + q\sin 4\lambda] - p}{1 - 2e^{4\lambda}\cos 4\lambda + e^{8\lambda}}$$
$$b_2(s) = \frac{e^{4\lambda}[p\sin 4\lambda - q\cos 4\lambda] + q}{1 - 2e^{4\lambda}\cos 4\lambda + e^{8\lambda}}$$
$$b_3(s) = e^{2\lambda}[-b_1(s)\cos 2\lambda + b_2(s)\sin 2\lambda]$$
$$b_4(s) = e^{2\lambda}[b_1(s)\sin 2\lambda + b_2(s)\cos 2\lambda]$$

$$p = c_1(s) + e^{2\lambda}[c_3(s)\cos 2\lambda + c_4(s)\sin 2\lambda]$$

$$q = c_2(s) + e^{2\lambda}[c_3(s)\sin 2\lambda - c_4(s)\cos 2\lambda]$$

We denote $c_i(s) = b_i(s) - a_i(s),\ i = 1,...,4$,

$$c_1(s) = \frac{e^{-\lambda(s+1)}}{8\lambda^3}[-\sin\lambda(s+1) + \cos\lambda(s+1)]$$

$$c_2(s) = \frac{e^{-\lambda(s+1)}}{8\lambda^3}[-\sin\lambda(s+1) - \cos\lambda(s+1)]$$

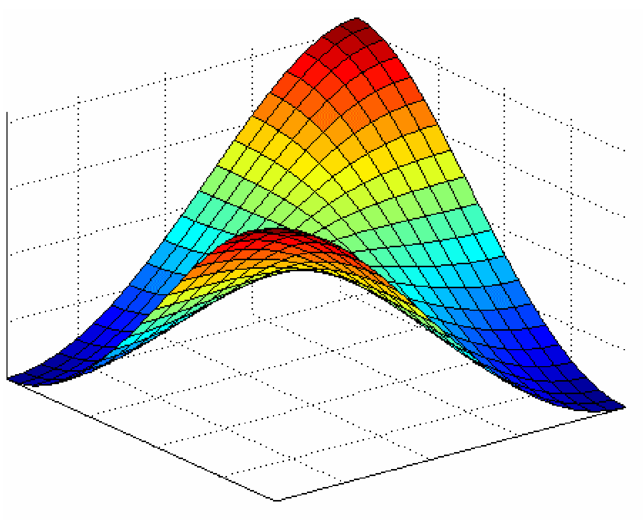
$$c_3(s) = \frac{e^{\lambda(s-1)}}{8\lambda^3}[\sin\lambda(s-1) + \cos\lambda(s-1)]$$

$$c_4(s) = \frac{e^{\lambda(s-1)}}{8\lambda^3}[-\sin\lambda(s-1) + \cos\lambda(s-1)]$$

and the coefficients $a_i(s)$ are determined as the differences

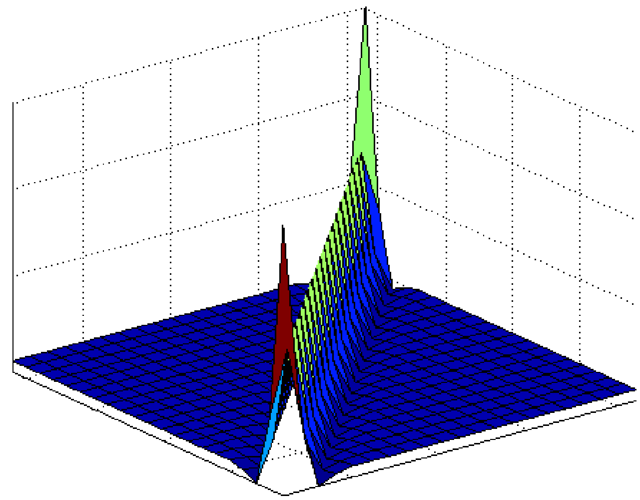
$$a_i(s) = b_i(s) - c_i(s),\ i = 1,...,4.$$

The graphs of the kernel for different values of λ are shown in Fig.1a and Fig.1b:



λ=0.5

Fig. 1a. Graph of the kernel $G_\lambda(x,s)$, $-1 < x < 1, -1 < s < 1$.



λ=15

Fig. 1b. Graph of the kernel $G_\lambda(x,s)$, $-1 < x < 1, -1 < s < 1$.

2.1 Comparison of filtering results by second and first order Tikhonov regularization

Tikhonov image filtering using first order derivative was considered in our previous paper [7]. The regularization functional in this case has the form

$$E_\alpha(u_\alpha) = \|u_\alpha - u_\delta\|_{L_2}^2 + \alpha\left\|\frac{d}{dx}u_\alpha\right\|_{L_2}^2.$$

In this section we compare the filtering properties of both methods. The comparison has been performed for test functions and medical images.

2.1.1 1D function smoothing

In order to test the performance of smoothing algorithms we chose the regularization parameter using discrepancy method.

Thus, parameter $\alpha = \alpha(\delta)$, is obtained as the solution of the following equation [3,4]:

$$\varphi(\alpha) = \int_{-1}^{1}(u_\alpha(x) - u_\delta(x))^2 dx = \delta^2.$$

The figure 2b illustrate the results of smoothing by both methods of a noisy function shown in figure 2a. The knowledge of the added noise enables us to evaluate δ and thus to find regularization parameter values. It can be seen that the second order method give better smoothing.

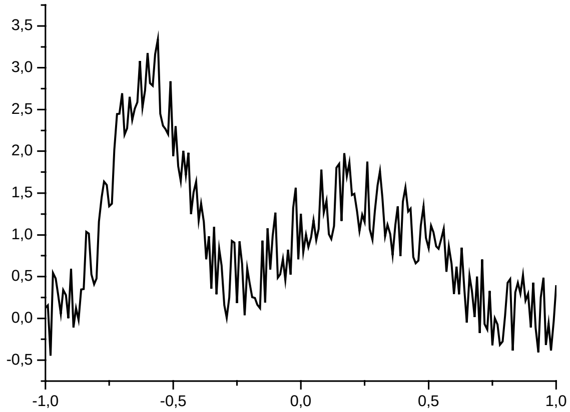


Fig. 2a. Source initial function with the added noise

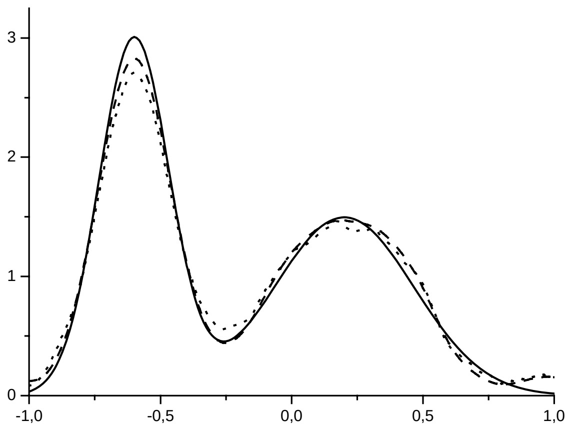


Fig. 2b. Function smoothing by first (---) and second (•••) order Tikhonov functional of the initial function (—).

2.1.2 Medical image filtering

In this section we compare the performance of the smoothing algorithms followed by Canny edge detection for noisy ultrasound heart picture (Fig.3a). Image smoothing was performed by consequent 1D filtering procedures.

The improvement in SNR (ISNR) metric was used to test the performance of image smoothing algorithms. The metric is given by

$$ISNR = 10 * \log_{10}\left(\frac{\sum_{i,j}[\overline{u}(i,j) - u_{\delta}(i,j)]^2}{\sum_{i,j}[\overline{u}(i,j) - u_{\alpha}(i,j)]^2}\right),$$

where $\overline{u}(i,j)$ and $u_{\delta}(i,j)$ are the original and degraded intensity components, respectively, and $u_{\alpha}(i,j)$ is the corresponding smoothed intensity field.

The comparison included the following steps:

- We choose regularization parameter of the first order Tikhonov functional that gives the best visible result and count ISNR value for the obtained filtered image.
- On the second step we choose regularization parameter of the second order Tikhonov functional giving the same ISNR value.
- Canny edge detection results are analyzed.

The figure 3 shows the result of image filtering by the second order Tikhonov functional. Zero-crossings of the second derivative obtained with Canny method for the filtering results of both methods are illustrated in figure 4 (the inverted intensity is proportional to the value of gradient modulus).

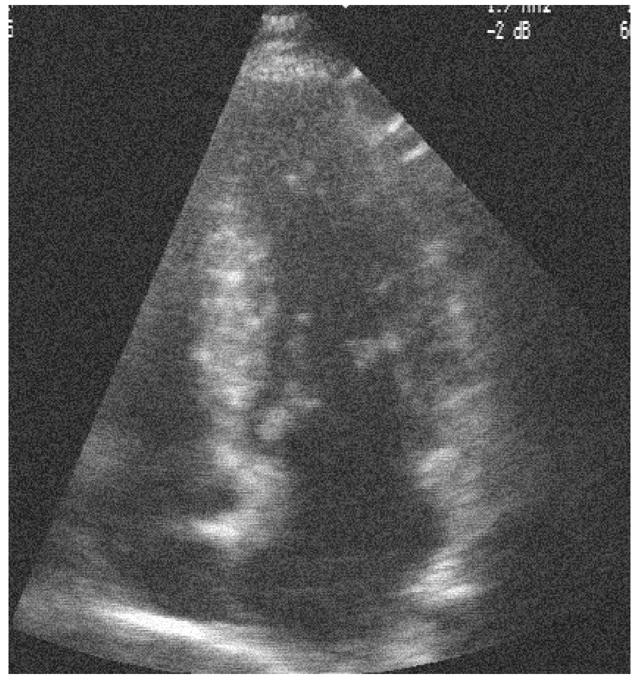


Fig. 3a. Source noisy image

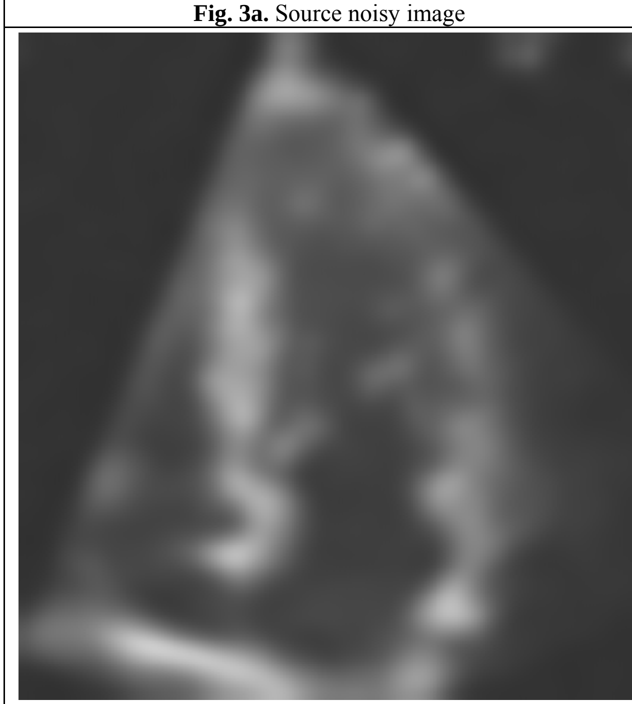


Fig. 3b. smoothed image

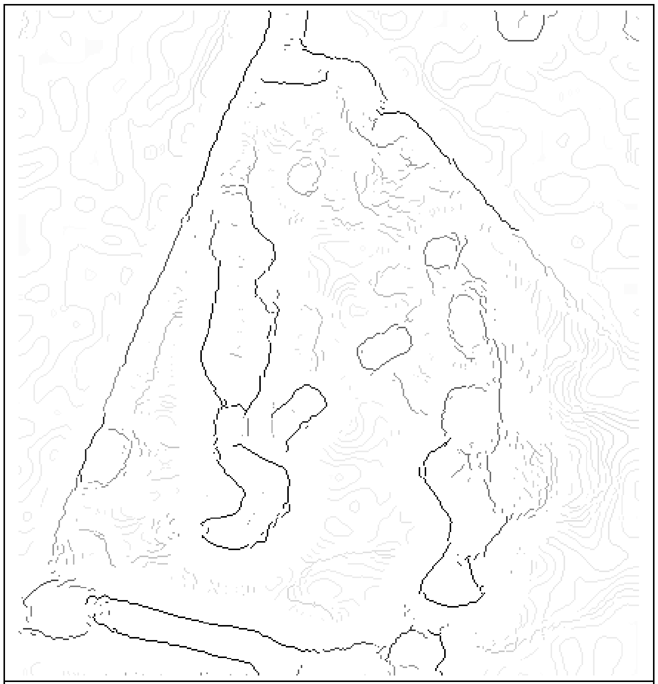


Fig. 4a. Edges of image. Filtering using second order Tikhonov functional

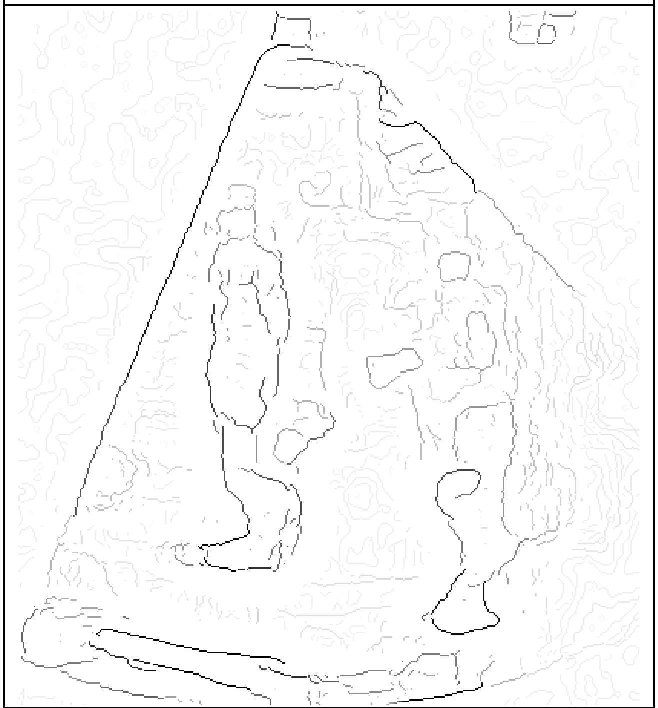


Fig. 4b. Edges of image. Filtering using first order Tikhonov functional

It can be seen that the new algorithm outperforms first order Tikhonov functional smoothing. Edges obtained after preliminary smoothing by second order Tikhonov functional and subsequent edge detection fit better the image segmentation task.

3. CONCLUSION

Second order Tikhonov regularization method for image filtering designed in the work looks very promising to be used in image segmentation tasks. The further investigations of the method are under progress.

ACKNOWLEDGEMENTS

This research was supported by AFOSR grant CRDF # RP0-1391-MO-03.

4. REFERENCES

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