3D Curve-Skeletons Extraction and Evaluation

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Abstract

A novel definition of the 3D curve-skeleton is presented. Many existing approaches to the problem can be formalized in the given definition. The main advantage of the presented mathematical model is that it allows strict quality assesment of the produced curve-skeleton. The definition is based on the usage of fat curves. A fat curve is a 3D object which allows to approximate tubular fragments of the shape. A set of fat curves is used to approximate the entire shape; such a set can be considered as a generalization of the 2D medial axis. An example algorithm which obtains curve-skeletons due to the given definition is also presented. The algorithm is robust and efficient.

Keywords: *curve-skeleton, medial axis, fat curve, shape analysis.*

1. INTRODUCTION

The medial axis, first introduced in [1], has been proved to be very useful for 2D shape analysis. The medial axis of a closed bounded set $\Omega \subset \mathbb{R}^2$ is the set of points having more than one closest point on the boundary; or, equivalently, the medial axis is the set of centers of the maximum inscribed in Ω circles. The medial axis is a graph embedded in $\mathbb{R}^2$. This graph emphasizes geometrical and topological properties of the shape Ω. Such graphs are usually called skeletons. There are efficient algorithms for 2D medial axis computation.

It would be natural to try use the same approach for 3D shape analysis. A medial axis of a 3D shape $\Omega \subset \mathbb{R}^3$ is a set of points having more than one closest point on the boundary. But such an object is not a graph since it may contain 2D sheets [2]. Those sheets may be very complex, and there are some methods which try to simplify the inner structure of the medial axis in 3d [3]. Therefore 3D medial axis is as difficult for the processing as the initial shape Ω. So there is a problem: how to define a skeleton of a 3D shape as a graph embedded in $\mathbb{R}^3$ so that this graph would have all of the advantages of the 2D medial axis?

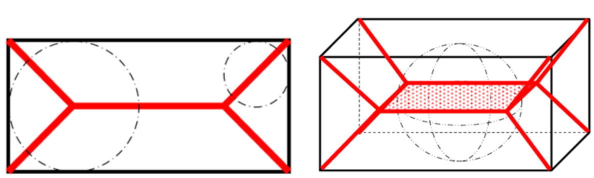

Figure 1: Medial axis of a 2D rectange and a 3D box.

Such graphs are called curve-skeletons. To date, there are lots of publications on curve-skeletons. However, unlike 2D case, where the strict mathematical definition of the medial axis was given decades ago, the definition of a 3D curve-skeleton still hasn't been presented. Usually, curve-skeleton is defined as the result of applying some algorithm to the 3D shape. There is no way to compare these algorithms with each other because their working principles and results may have totally different nature. It's very difficult to evaluate the quality of the skeletons produced by those algorithms, since there is no formal criterion for such an evaluation. There's only visual evaluation, which is subjective and not mathematical at all.

In [4][5] the authors presented the classification of curve-skeleton algorithms. The curve-skeleton is intuitively defined as a 1D thinning of the 3D object. The authors also made a list of possible properties of a curve-skeleton. Some of these properties are strict (for example, topological equivalency between curve-skeleton and the original shape), others are intuitive and should be formalized (centeredness of the skeleton and possibility of reconstruction of the original 3D object). Almost every published algorithm computes skeletons which have some of these properties. In such cases, these properties are considered to be advantages of the algorithm.

One of the popular approaches is based on the thinning of voxel images. Such thinning may be done directly (deleting boundary voxels step-by-step, [6] [7]) or with the distance function [8]. The skeletons produced by such methods are not continuous but discrete objects. Algorithms of this class are not universal because they're applicable to the voxel images only. Finally, there is no mathematical criterion to evaluate and compare different techniques of thinning.

It seems natural to try to extract 1D curve-skeleton from the 2D medial axis. The medial axis itself is a very complicated object, which consists of quadratic surfaces, so it's usually replaced by some approximation. However, extraction of 1D piece from the medial axis usually based on some successfully found heuristic. For example, in [9] such an extraction is done with the help of the geodesics on the boundary surface. As in the previous case, there's no strict criterion for evaluation and comparison of various heuristics of a 1D curve-skeleton extraction.

There are some other techniques used to compute curve-skeleton, such as usage of optimal cut planes [10] or physical interpretation of the problem [11]. However, these methods are also successfully found heuristics which allow to compute some object visually corresponding to the human idea of the curve-skeleton. And again, formal mathematical evaluation of these algorithms doesn't seem to be possible.

In this paper, a strict definition of the curve-skeleton is given. The model being proposed

1. allows to evaluate the correspondence between the curve-skeleton and the original object;
2. approximates the given shape with a fixed precision;
3. doesn't depend on the type of the shape description (polygonal model, voxel image or point cloud).

Also, an algorithm which computes the curve-skeleton according to the definition, is presented.

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2. DEFINITIONS

Let $\mathscr{C}$ be a set of smooth curves in $\mathbb{R}^3$. For every curve $c \in \mathscr{C}$, there is a set $\mathscr{R}_c$ of continuous non-negative functions defined on c.

Definition 1 *A fat curve is a pair (c, r), where $c \in \mathscr{C}, r \in \mathscr{R}_c$. The curve c is said to be an axis of the fat curve, and the function r is its radial function.*

Definition 2 *An image of the fat curve (c, r) is a set of points*

$$I(c, r) = \{\mathbf{x} \in \mathbb{R}^3 | \exists \mathbf{y} \in c : \rho(\mathbf{x}, \mathbf{y}) \leq r(\mathbf{y})\}. \quad (1)$$

Definition 3 *A boundary of the fat curve (c, r) is a set of points*

$$\partial I(c, r) = \{\mathbf{x} \in I(c, r) | \forall \mathbf{y} \in c : \rho(\mathbf{x}, \mathbf{y}) \geq r(\mathbf{y})\}. \quad (2)$$

The fat curve is an object which is very convenient to approximate tubular 3D shapes (see Fig. 2).

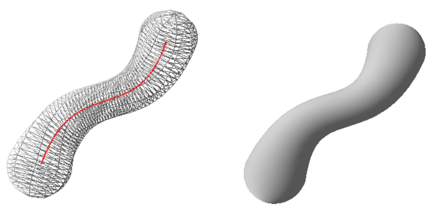


Figure 2: Fat curve.

Definition 4 *Let C be a set of fat curves such that axis of these fat curves intersect each other in their endpoints only. A fat graph F over a set C is a graph whose edges are fat curves from C and vertices are endpoints of their axis.*

Definition 5 *A boundary ∂F of a fat graph F is an union of boundaries of the fat curves composing F.*

Let $\mathscr{F}_{\mathscr{C}}$ be a set of all possible fat graphs.

Consider an embedded in $\mathbb{R}^3$ connected 3D manifold Ω with the boundary $\partial\Omega$. We'll approximate Ω with some fat graph.

Definition 6 *A distance between the point $\mathbf{x} \in \mathbb{R}^3$ and the fat graph F is a distance between $\mathbf{x}$ and the closest point on $\mathscr{F}_{\mathscr{C}}$'s boundary:*

$$\rho(\mathbf{x}, G) = \min_{\mathbf{y} \in \partial F} \rho(\mathbf{x}, \mathbf{y}). \quad (3)$$

Definition 7 *A distance between a manifold Ω and a fat graph F is a value*

$$\varepsilon(\Omega, F) = \int_{\mathbf{x} \in \partial\Omega} \rho^2(\mathbf{x}, F) dS. \quad (4)$$

Approximation quality can be evaluated by two values: distance $\varepsilon(\Omega, F)$ and complexity of the fat graph F. A complexity of a fat graph can be evaluated as

1. sum of lengths of axis of fat curves composing the fat graph;
2. number of the fat curves.

If the set $\mathscr{C}$ is rather wide, it's better to use the first criterion in order to avoid too crooked curves. However, if $\mathscr{C}$ is narrow and doesn't contain such curves, the second criterion can be used since it's very simple.

In these terms, the problem of approximation with a fat graph can be defined as following

1. compute an approximation with the smallest possible $\varepsilon(\Omega, F)$ and a fixed fat graph complexity;
2. compute an approximation with the least possible complexity and
$$\varepsilon(\Omega, F) < \varepsilon_0, \quad (5)$$
where ε_0 is a fixed value.

The fat graph can also be defined for planar curves. 2D medial axis would be an example of such a graph if we define the radial function at a point $\mathbf{x}$ equal to the distance from $\mathbf{x}$ to the boundary. Image of this special graph coincides with the whole shape, so its approximation error is zero.

3. IMPLEMENTATION

The main issue which wasn't discussed in the previous chapter but seems to be very important in the practical implementations of the method is how to choose the first approximation of the skeleton. It's possible to use any existing algorithm which produces curve-skeletons. But the proposed scheme has the advantage that the first approximation of the skeleton doesn't have to be very nice and accurate. It can be easily fitted into the shape afterwards using numerical methods.

That means that we can use some algorithm which is inaccurate but very fast, hoping to improve it during the fat graph fitting. One way to do so is to use the 2D medial axis of some Ω's planar projection. There're some facts in favor of this decision.

- It seems that medial information plays the key role in the human vision and visual perception[5]. But the human vision is planar, so if some 3D graph feels to be a good curve-skeleton, its projection would be also considered as a graph which is very close to the 2D medial axis of the object.
- 2D medial axis is a well-defined and examined object. There are fast and robust algorithms for 2D skeletonization.
- As mentioned above, the 2D medial axis can be considered as a fat graph which has zero approximation error. It's possible to try to bring this property in 3D as close as possible.
- 2D medial axis of a polygonal shape consists of smooth curves of degree 1 and 2, which can be fitted in the manner described above.

An example of the object and its planar projection is shown in the Fig. 3.

Each knot of the 2D skeleton is a projection of at least 2 points on the surface (see Fig. 4). A maximal inscribed ball tangent to the surface $\partial\Omega$ at those two points is a good mapping of the knot into the 3D space.

The main problem of this method is the possibility of occlusions. For example, if one of the legs of the horse in Fig. 3 was occluded by another one, the usage of this particular projection would lead to an incorrect result. For some models it's possible to find a projection which gives no occlusions. The incorrect projection with occlusions produces significant approximation error. But for some

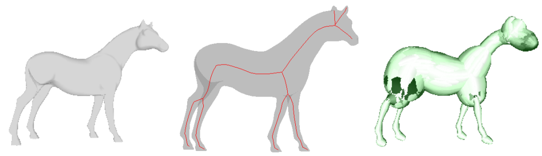


Figure 3: A model of a horse (left), its orthogonal planar projection with the 2D medial axis (center) and the approximating fat graph (right).

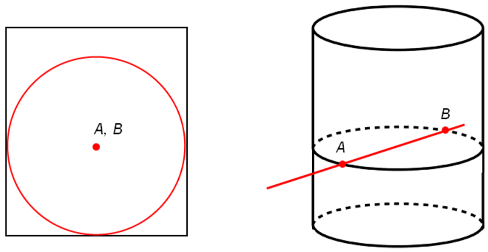


Figure 4: 2D projection (left) of a 3D cylinder (right); center of the maximum inscribed circle is a projection of points A, B.

shapes it's impossible (or very difficult) to find a good projection with no serious occlusions. This problem is solved by the preliminary segmentation. The shape is divided into tubular segments. Each segments is approximated with its own fat graph produced by some particular planar projection. Finally, all these partial fat graphs are joined into one.

The segmentation is defined by a set of points

$$Q_s = \{\mathbf{q_1}, \dots, \mathbf{q_s}\}, \mathbf{q_i} \in \partial\Omega. \tag{6}$$

Let ρ_Ω be a geodesic distance on the surface $\partial\Omega$. Then each segment S_i is defined as a set of points closest to $\mathbf{q_i}$:

$$S_i = \{\mathbf{x} \in \partial\Omega | \forall k, 1 \leq k \leq s, \rho_\Omega(\mathbf{x}, \mathbf{q_i}) \leq \rho_\Omega(\mathbf{x}, \mathbf{q_k})\}. \tag{7}$$

An example is shown on the Fig. 5.

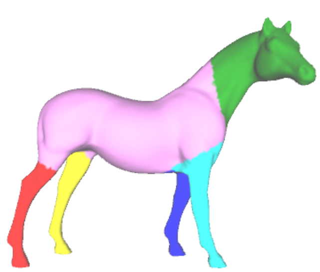


Figure 5: Segmentation of the model.

If the segmentation based on the set Q_s is not detailed enough, we can replace it with a new one

$$Q_{s+1} = Q_s \cup \{\mathbf{q_{s+1}}\}, \mathbf{q_{s+1}} = \arg\max_{\mathbf{x} \in \partial\Omega} \rho_\Omega(\mathbf{x}, Q_s), \tag{8}$$

where

$$\rho_\Omega(\mathbf{x}, Q_s) = \min_{1 \leq i \leq s} \rho_\Omega(\mathbf{x}, \mathbf{q_i}). \tag{9}$$

The segmentation process starts with the set Q_2 which consists of two points which are most distant from each other.

The short summary of this chapter gives us the following scheme of the algorithm.

1. Segmentation of the model.
2. Choosing the best 2D projection for each segment (the word "best" means that this projection leads to the least value of the approximation error on the next step).
3. Computation of the approximating fat graph for each segment using the planar skeleton of the projection obtained on the previous step.
4. Join all of the fat graphs into one and final fitting using the numerical methods.

4. EXPERIMENTS

The described algorithm was successfully implemented. As mentioned above, it's impossible to compare the quality of the skeletons produced by other methods, since there has been no numerical criterion for evaluation of the difference between the curve-skeleton and the given shape. We'll prove the capacity of our approach in the way that is common in the literature on the curve-skeletons, which is based on the visual evaluation and experimental proof of the claimed properties.

First of all, we demonstrate the examples of curve-skeletons produced by the described algorithm (see Fig. 6). 3D models which have been chosen for the experiment are widely used to evaluate various computer geometry algorithms, so they're appropriate for the visual comparison with other methods.

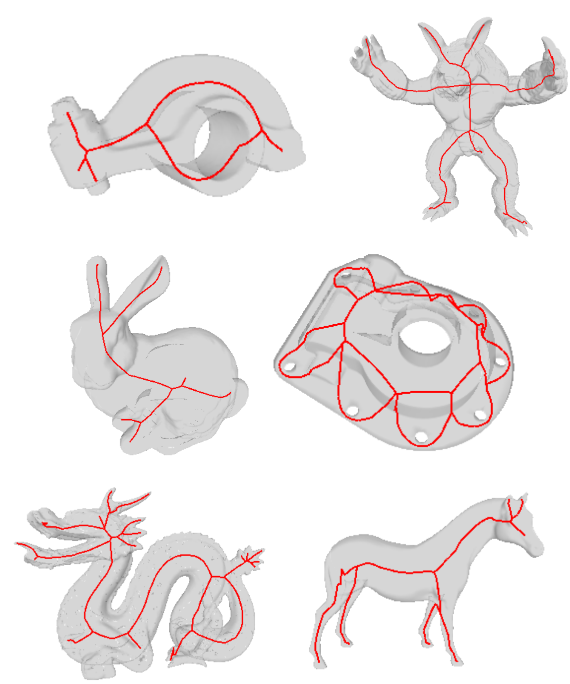
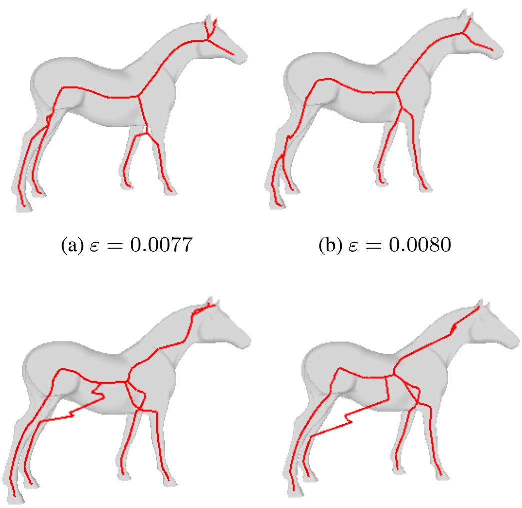


Figure 6: Examples.

It's useful to discuss the properties listed in [4]. Homotopy equivalence between the curve-skeleton and the shape is not guaranteed, since the fat graph with relatively large amount of edges approximates wide non-tubular fragments of the shape with a loop consisting of a pair of edges. However, this property can be easily provided by more strict requirements for the topological class. Invariance under isometric transformations (i.e. transformations in which the distances between points are preserved) is obvious. The possibility of reconstruction of the original shape is provided by the definition of a fat graph: the image of the fat graph is a 3D manifold which approximates the original object with a known precision. The reliability (which means that every boundary point is visible from at least one curve-skeleton location) is not guaranteed, but it's achieved when the fat graph has enough edges. The robustness is implied by the robustness of the function $\varepsilon(\Omega, F)$.

In order to justify the meaningfulness of the function $\varepsilon(\Omega, F)$, which is the core part of the described method, we've prepared a number of various curve-skeletons of the same object. These skeletons were made without any fitting and with badly tuned parameters of the algorithm. The curve-skeletons and their corresponding approximation error values are shown on the Fig. 7. It's pretty obvious that the greater the value of $\varepsilon(\Omega, F)$, the worse the visual quality of the produced curve-skeleton. That implies that the proposed definition is not only theoretically grounded but also has some practical utility.



(a) $\varepsilon = 0.0077$ (b) $\varepsilon = 0.0080$

(c) $\varepsilon = 0.0082$ (d) $\varepsilon = 0.0089$

Figure 7: Curve-skeletons of the horse with different approximation error.

5. CONCLUSION

In the paper, a new mathematical model for curve-skeleton formalization was presented. This model allows to compare and research various approaches for the 3D skeletonization. Also, a new algorithm for skeletonization was described, implemented and discussed. The further research involves the following issues:

- elaboration of the model, in particular, approximation evaluation via Hausdorff metric;
- further development of the algorithm, better selection of the first approximation and formalization of the iterative fitting.

6. REFERENCES

[1] Harry Blum, "A Transformation for Extracting New Descriptors of Shape," in *Models for the Perception of Speech and Visual Form*, Weiant Wathen-Dunn, Ed., pp. 362–380. MIT Press, Cambridge, 1967.

[2] Peter Giblin and Benjamin B. Kimia, "A formal classification of 3d medial axis points and their local geometry," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 26, pp. 238–251, January 2004.

[3] Ming-Ching Chang and Benjamin B. Kimia, "Regularizing 3D medial axis using medial scaffold transforms," in *Proceedings of the IEEE Conference on Computer Vision and Pattern Recognition*. 2008, p. Accepted, IEEE Computer Society.

[4] Nicu D. Cornea and Deborah Silver, "Curve-skeleton properties, applications, and algorithms," *IEEE Transactions on Visualization and Computer Graphics*, vol. 13, pp. 530–548, 2007.

[5] Kaleem Siddiqi and Stephen Pizer, *Medial Representations: Mathematics, Algorithms and Applications*, Springer Publishing Company, Incorporated, 1st edition, 2008.

[6] Kálmán Palágyi and Attila Kuba, "A parallel 12-subiteration 3d thinning algorithm to extract medial lines," in *Proceedings of the 7th International Conference on Computer Analysis of Images and Patterns*, London, UK, 1997, CAIP '97, pp. 400–407, Springer-Verlag.

[7] Yu-Shuen Wang and Tong-Yee Lee, "Curve-skeleton extraction using iterative least squares optimization," *IEEE Transactions on Visualization and Computer Graphics*, vol. 14, pp. 926–936, July 2008.

[8] Alexandru Telea and Jarke J. van Wijk, "An augmented fast marching method for computing skeletons and centerlines," in *Proceedings of the symposium on Data Visualisation 2002*, Aire-la-Ville, Switzerland, Switzerland, 2002, VISSYM '02, pp. 251–ff, Eurographics Association.

[9] Tamal K. Dey and Jian Sun, "Defining and computing curve-skeletons with medial geodesic function," in *Proceedings of the fourth Eurographics symposium on Geometry processing*, Aire-la-Ville, Switzerland, Switzerland, 2006, pp. 143–152, Eurographics Association.

[10] Andrea Tagliasacchi, Hao Zhang, and Daniel Cohen-Or, "Curve skeleton extraction from incomplete point cloud," *ACM Trans. Graph.*, vol. 28, pp. 71:1–71:9, July 2009.

[11] Jen-Hi Chuang, Chi-Hao Tsai, and Min-Chi Ko, "Skeletonization of three-dimensional object using generalized potential field," *IEEE Trans. Pattern Anal. Mach. Intell.*, vol. 22, pp. 1241–1251, November 2000.

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