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Projective reconstruction from curve correspondences in uncalibrated views

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Abstract

This paper presents algorithms for stereo reconstruction from uncalibrated binocular stereo views using correspondences of curves between the two images. We mainly consider curves which are projections of planar 3-D curves although some of our methods work without this restriction as well. First a brief description of reconstruction algorithms based on point correspondences is given. Then we introduce a coordinate system which is invariant to the unknown camera parameters. Compared to similar constructions reported in the literature our method has the advantage that no restrictions concerning the relative position of the cameras (e.g. parallel optical axes) are necessary. Using this coordinate system we show how to reduce the reconstruction from conic correspondences to the point based case. Next the case of general curves is considered. By applying a simple tangent construction we obtain for a given pair of corresponding curves at least two pairs of matching curve points. By fitting at these points an osculating conic we get pairs of matching conics which reduces the problem to conics based stereo reconstruction. All these algorithms rely on the assumption that the epipolar geometry (encoded in the fundamental matrix) is known. We discuss how to recover the epipolar geometry from conic correspondences. The appendix contains some useful results concerning normal forms of pairs of conics.

Chapter 1

Introduction

Research in the last few years (e.g. [4, 6, 8, 23, 17, 18, 2, 13]) has demonstrated that many stereo based 3D vision tasks like obstacle avoidance, object recognition or motion and structure estimation can be accomplished with an uncalibrated stereo rig. Obviously this is advantageous since camera calibration is an awkward process which cannot always be performed reliably (e.g. for autonomous vehicles).

Basically all stereo techniques aim at inferring 3D information from correspondences established between appropriate features in the two images. These features must be extracted in a preprocessing step and the correspondences must be established. Even if fully calibrated cameras are assumed many techniques for 3D reconstruction are restricted to point or line features. In this case reconstruction can be achieved by conventional triangulation techniques (cf. [5] and the extensive bibliography cited there).

In [21, 11] reconstruction using correspondences of planar conics is discussed for the case of calibrated stereo. [12] extends these techniques to the reconstruction of quadric surfaces from the occluding contour, whereas [22] puts the focus on robustness issues in conic based reconstruction from fully calibrated views.

Most authors addressing the problem of uncalibrated stereo use discrete points as features (e.g. [4, 13, 6]). In [7, 8] the case of line features is considered. This is a progress since lines are easier to detect than points. Furthermore establishing the correspondences is facilitated since there are normally fewer lines than points in an image.

In this paper we consider the problem of 3D reconstruction from uncalibrated stereo views using curve correspondences. First a brief description of reconstruction algorithms based on point correspondences is given. Then we introduce a coordinate system which is invariant to the unknown camera parameters. Compared to similar constructions reported in the literature our method has the advantage that no restrictions concerning the relative position of the cameras (e.g. parallel optical axes) are necessary. Using this coordinate system we show how to reduce the reconstruction from conic correspondences to the point based case. Next the case of general curves is considered. By applying a simple tangent construction we obtain for a given pair of corresponding curves at least two pairs of matching curve points. By fitting at these points an osculating conic we get pairs of corresponding conics. This reduces the problem to conics based stereo reconstruction from uncalibrated views. Due to the fact that the stereo setup is uncalibrated no complete 3D reconstruction can be achieved. If no a-priori knowledge about the scene (e.g. coplanarity of curves) is available then a reconstruction up to a projectivity can be achieved. This is in analogy with results for the point based case ([4, 6]).

For the aforementioned tangent construction we need the epipolar geometry (encoded in the fundamental matrix) of the stereo rig. We present a method for determining the fundamental matrix and the coordinates of the epipoles from the correspondences of a pair of planar conics, i.e. in every image we can identify two conics and correspondences between the conics can be established. These image conics are assumed to be the projections of two planar conics in 3D space.

In the appendix we summarize basic facts from projective geometry and prove some new results concerning normal forms of pairs of conics.

Chapter 2

Basic Concepts

2.1 Statement of the problem

We consider a threedimensional scene consisting of a set of curves. We take a stereo pair of images with an uncalibrated stereo rig, i.e. we have no information about the relative displacement of the two cameras (external parameters) and the internal camera parameters (focal lengths, principal points, etc..). The question is addressed what kind of 3D reconstruction can be achieved from such a stereo pair under the assumption that correspondences between a subset of curves in the images can be established. A typical setup is shown in Figure 2.1.

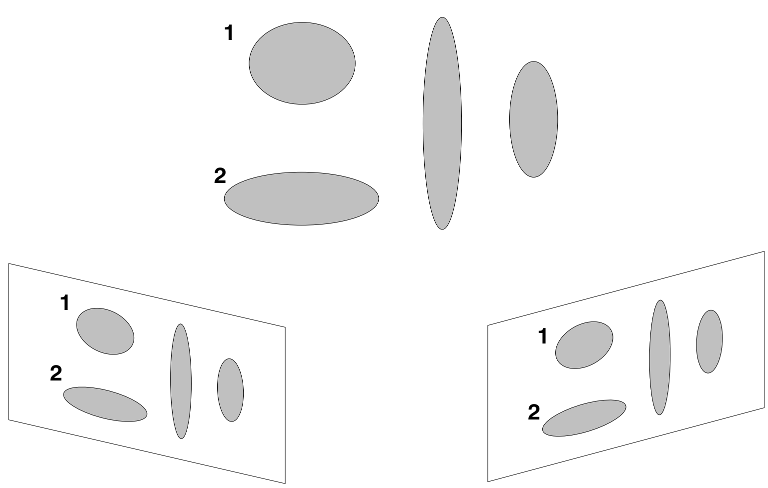

Figure 2.1: A typical setup for stereo reconstruction from curve correspondences. The established correspondences are indicated by the numbers $1, 2$.

First we assume that the epipolar geometry of the stereo rig is known and show how we can achieve a 3D reconstruction up to projective transformation. Afterwards we discuss how to recover the epipolar geometry from correspondences of planar conics (algebraic curves of second order), i.e. we assume that the image curves are projections of flat curves in 3D.

2.2 Terminology

In this section we fix our terminology and introduce some basic concepts from projective geometry. We model the camera planes as two-dimensional complex projective spaces $\mathbb{P}^2$ (using complex instead of real spaces is not an essential point but it facilitates the subsequent mathematical analysis). The elements of $\mathbb{P}^2$ are homogeneous three vectors $\mathbf{x} = (x_1, x_2, x_3)^T$, i.e. all vectors of the form $\lambda\mathbf{x} = (\lambda x_1, \lambda x_2, \lambda x_3)^t, \lambda \in \mathbb{C} \setminus \{0\}$ represent the same point in $\mathbb{P}^2$. Vectors $\mathbf{x}$ are written as column vectors. In order to indicate that a pair of points in the two images corresponds to the same point in 3D we use a prime, i.e. $\mathbf{x}$, $\mathbf{x}'$ are image coordiantes from the left and right camera respectively which are projections of one 3D point. We model 3-D space as threedimensional projective space $\mathbb{P}^3$ and denote the homogeneous coordinates for points in $\mathbb{P}^3$ by uppercase vectors like $\mathbf{X} = (X_1, X_2, X_3, X_4)$. Furthermore we denote by $\langle \mathbf{X}_1, \mathbf{X}_2 \rangle$ the line spanned by the two points $\mathbf{X}_1, \mathbf{X}_2$ and by $\langle \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 \rangle$ the plane spanned by the three points $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$.

Let $M(\mathbb{P}^2)$ be the set of all 3×3 matrices with complex entries which are defined up to a common scale factor, i.e. two matrices $\underline{\mathbf{A}}, \underline{\mathbf{B}}$ represent the same element of $M(\mathbb{P}^2)$ if a $\lambda \in \mathbb{C} \setminus \{0\}$ exists with $\underline{\mathbf{A}} = \lambda \underline{\mathbf{B}}$. By $GL(\mathbb{P}^2)$ we denote the set of all matrices $\underline{\mathbf{A}} \in M(\mathbb{P}^2)$ with $\det \underline{\mathbf{A}} \neq 0$. A matrix $\underline{\mathbf{A}} \in M(\mathbb{P}^2)$ is called symmetric if it is left unaltered under matrix transposition, $\underline{\mathbf{A}}^T = \underline{\mathbf{A}}$.

For a symmetric matrix $\underline{\mathbf{C}} \in M(\mathbb{P}^2)$ the conic $\mathcal{C}$ is defined as the following subset of $\mathbb{P}^2$:

$$\mathcal{C} := \{\mathbf{x} \in \mathbb{P}^2 \mid \mathbf{x}^T \underline{\mathbf{C}} \mathbf{x} = 0\}. \tag{2.1}$$

A conic $\mathcal{C}$ is called nondegenerate if the describing matrix $\underline{\mathbf{C}}$ is nonsingular ($\underline{\mathbf{C}} \in GL(\mathbb{P}^2)$), otherwise $\mathcal{C}$ is called degenerate. For every matrix $\underline{\mathbf{A}} \in M(\mathbb{P}^2)$ and every symmetric matrix $\underline{\mathbf{C}} \in M(\mathbb{P}^2)$ it is easy to see that the matrix $\underline{\mathbf{A}}^T \underline{\mathbf{C}}\, \underline{\mathbf{A}}$ is also symmetric. Therefore $\underline{\mathbf{C}} \rightarrow \underline{\mathbf{A}}^T \underline{\mathbf{C}}\, \underline{\mathbf{A}}$ maps conics to conics.

Apart from points $\mathbf{x} \in \mathbb{P}^2$ and conics we have also to work with lines. For a given vector $\mathbf{l} = (l_1, l_2, l_3)^T$ the set of points $\mathbf{x} \in \mathbb{P}^2$ which are solutions of the equation $\mathbf{x}^T \mathbf{l} = 0$ form a line in $\mathbb{P}^2$. We call $\mathbf{l}$ a coordiante vector for that line.

2.3 Summary of point based algorithms

We denote 3D points by uppercase letters in boldface, e.g. $\mathbf{M}$, whereas the projections of a 3D point $\mathbf{M}$ onto the two retinas are denoted by $\mathbf{m}$ and $\mathbf{m}'$ respectively. When assuming the usual pinhole model for the two cameras the relationship between 3D and 2D can be described by two projection matrices $\underline{\mathbf{P}}, \underline{\mathbf{P}}^{'}$. This are 4×3 matrices so that for any 3D point $\mathbf{M}$

$$\mathbf{m} = \underline{\mathbf{P}}\mathbf{M} \text{ and } \mathbf{m}^{'} = \underline{\mathbf{P}}^{'}\mathbf{M}. \tag{2.2}$$

We consider three different coordinate systems. One for the 3D description and the other two for describing the image planes. Futhermore we assume that five 3D points in general position (i.e. no four coplanar) are given. We denote these points by $\mathbf{M}_i$, $1 \leq i \leq 5$ and the projections by $\mathbf{m}_i, \mathbf{m}_i^{'}$, $1 \leq i \leq 5$. Since we are assuming uncalibrated cameras neither the 3D coordiante system, the image coordinate systems nor their mutual relations are fixed. This can be rephrased in two different ways. On the one hand we can say that our intended 'reconstruction' must be invariant with respect to transformations of these three coordinate systems. On the other hand we can say that we assume that the coordinate systems have some standard forms so that the description of the vectors $\mathbf{M}_i$, $\mathbf{m}_i$, $\mathbf{m}_i^{'}$ is especially simple. We define $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. Then we can introduce a 3D and two 2D coordinate systems so that

$$\begin{aligned}
\mathbf{M}_i &= (\delta_{i1}, \delta_{i2}, \delta_{i3}, \delta_{i4})^T \ \forall 1 \leq i \leq 4 \\
\mathbf{M}_5 &= (1,1,1,1)^T \\
\mathbf{m}_i = \mathbf{m}_i^{'} &= (\delta_{i1}, \delta_{i2}, \delta_{i3})^T \ \forall 1 \leq i \leq 3 \\
\mathbf{m}_4 = \mathbf{m}_4^{'} &= (1,1,1)^T.
\end{aligned} \tag{2.3}$$

The remaining points $\mathbf{m}_5$, $\mathbf{m}_5^{'}$ can be expressed in terms of appropriate cross ratios of $\mathbf{m}_i$, $\mathbf{m}_i^{'}$, $1 \leq i \leq 4$. Assuming the knowledge of the epipoles $\mathbf{e}, \mathbf{e}^{'}$ it is shown in [4] how to determine from equations (2.2) and the standard forms in equations (2.3) the projection matrices $\underline{\mathbf{P}}, \underline{\mathbf{P}}^{'}$. Given the projection matrices and a pair of matching points $\mathbf{m}$, $\mathbf{m}^{'}$ it is not difficult to compute the 3D coordinates $\mathbf{M}$ by using the equations (2.2).

The important point to note is that the fact that the stereo rig is uncalibrated is exploited to introduce 'standard coordinate systems' for the analytical description. This removes all ambiguities due to unknown internal and external camera parameters.

2.4 The fundamental matrix

In order to apply the methods described in section 2.3 it is necessary to determine the coordinates of the epipoles. For this purpose it is advantageous to use the fundamental matrix $\underline{\mathbf{F}}$ which is the most important tool for the reconstruction from uncalibrated views (cf. [4]). This matrix is a generalization of the essential matrix introduced by Longuet-Higgens in [10] for computing the relative camera displacement from image corespondences obtained by calibrated cameras. For a given binocular stereo rig the fundamental matrix $\underline{\mathbf{F}}$ is a 3×3 matrix of rank 2 with the following properties ([5]):

a) if $\mathbf{x}$, $\mathbf{x}^{'}$ are a pair of matching points, then

$$\mathbf{x}^{'T}\underline{\mathbf{F}}\mathbf{x} = 0. \tag{2.4}$$

b) We denote by $\mathbf{e}, \mathbf{e}^{'}$ the coordiantes of the epipoles in the two camera planes. Then

$$\left(\mathbf{e}^{'T}\underline{\mathbf{F}}\right)^{T} = \underline{\mathbf{F}}\mathbf{e} = \mathbf{0}. \tag{2.5}$$

$\mathbf{0}$ is the vector $(0, 0, 0)^T$.

If we set $\mathbf{l} = \underline{\mathbf{F}}\mathbf{x}$ then equation (2.4) reduces to $\mathbf{x}^{'T}\mathbf{l} = 0$ which can be interpreted as the equation of a line in the primed coordinate system. Therefore the fundamental matrix maps points from one image to lines in the other image. All these lines which are called epipolar lines go through the epipole $\mathbf{e}^{'}$. A reformulation of this fact is to say that for a point $\mathbf{x}$ the corresponding point $\mathbf{x}^{'}$ must be on the epipolar line $\underline{\mathbf{F}}\mathbf{x}$.

By using the equation (2.4) it is possible to determine $\underline{\mathbf{F}}$ from eight point matches. Once $\underline{\mathbf{F}}$ is known the epipoles can be determined via the equations (2.5).

Chapter 3

Reconstruction from curve correspondences and known epipoles

3.1 Invariant Coordinates

In this section we build a coordinate system which is invariant to the unknown camera parameters. We show the relation between these coordinates and homogeneous 3-D projective coordinates, for a general configuration of cameras. The derivation was inspired by the work in [4, 13, 6, 1], but to the best of our knowledge it is more general than previous derivations.

We first build an invariant coordinate system in 3-D. We assume for the moment that the positions of the two camera centers $\mathbf{O}, \mathbf{O}'$ in 3-D are known. We also know the positions of some three non-collinear points $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$. It is assumed that the five points $\mathbf{O}, \mathbf{O}', \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are in general position (i.e. no four coplanar). These five points are used as a basis for a projective invariant coordinate system in 3-D space. This system can be build as follows.

First create the three planes which have the line $\mathbf{OO}'$ in common and pass through the three $\mathbf{X}_i$. Namely, we have the three planes

$$\langle \mathbf{O}, \mathbf{O}', \mathbf{X}_1 \rangle,\ \langle \mathbf{O}, \mathbf{O}', \mathbf{X}_2 \rangle,\ \langle \mathbf{O}, \mathbf{O}', \mathbf{X}_3 \rangle.$$

Given some other 3-D point $\mathbf{X}$, we can build a fourth plane containing $\langle \mathbf{O}, \mathbf{O}' \rangle$, namely $\langle \mathbf{O}, \mathbf{O}', \mathbf{X} \rangle$. We now have a set of four planes containing the same line $\langle \mathbf{O}, \mathbf{O}' \rangle$, so we can calculate their cross ratio. We take this cross ratio as the first invariant coordinate P_1 of the point $\mathbf{X}$. The other two invariant coordinates P_2, P_3 can be defined in a similar way by using sets of planes having a different common line. P_2, for example, is determined by choosing $\langle \mathbf{O}, \mathbf{X}_1 \rangle$ as the common line, building the four planes that contain this line and the remaining four points $\mathbf{O}', \mathbf{X}_2, \mathbf{X}_3, \mathbf{X}$, and finding the cross ratio of these planes. P_3 is determined in the same way by replacing the common line $\langle \mathbf{O}, \mathbf{X}_1 \rangle$ above with $\langle \mathbf{O}', \mathbf{X}_1 \rangle$.

Since the coordinates were defined by cross ratios, they are invariant to a 3-D projective transformation of the $\mathbf{X}$ coordinates. It is well known ([16], p. 127) that the cross ratios P_i as defined above, can be used as homogeneous coordinates in the corresponding projective space. That is, we can write a four-component vector $\mathbf{P}$ as

$$\mathbf{P} = (P_1, P_2, P_3, 1)^T.$$

The invariant homogeneous coordinates $\mathbf{P}$ are related to the Cartesian homogeneous coordinates $\mathbf{X} = \lambda(X, Y, Z, 1)^T$ by a projective transformation, namely a 4×4 linear matrix:

$$\mathbf{P} = \underline{\mathbf{A}}\mathbf{X}.$$

The transformation $\underline{\mathbf{A}}$ is determined by the coordinates of the five reference (basis) points, $\mathbf{O}, \mathbf{O}', \mathbf{X}_i$, in the two coordinate systems.

The interesting fact here is that the vector $\mathbf{P}$ can be calculated from a stereo pair of uncalibrated cameras, given the projections of the five basis points. Therefore, the $\mathbf{X}$ coordinates can be reconstructed from uncalibrated cameras only up to a projective transformation $\underline{\mathbf{A}}$. This transformation contains camera parameters that cannot be measured from the image, such as the 3-D positions of the camera centers. Since $\mathbf{P}$ is independent of $\underline{\mathbf{A}}$, it is invariant to such unknown camera parameters.

We now want to find the relation between the 3-D coordinates $\mathbf{P}$ above and the 2-D coordinates of the given image pair. The projections of $\mathbf{X}$ in the two images are denoted by $\mathbf{x}$, $\mathbf{x}'$ respectively. The camera centers $\mathbf{O}, \mathbf{O}'$ are projected as $\mathbf{e}'$, $\mathbf{e}$ (the epipoles) respectively. To deal with the first coordinate P_1, we look at the projection of the four planes that were used to define it. These four planes $\langle \mathbf{O}, \mathbf{O}', \mathbf{X}_i \rangle$, $\langle \mathbf{O}, \mathbf{O}', \mathbf{X} \rangle$ are projected on the two images as lines $\langle \mathbf{e}, \mathbf{x}_i \rangle$, $\langle \mathbf{e}, \mathbf{x} \rangle$ and $\langle \mathbf{e}', \mathbf{x}'_i \rangle$, $\langle \mathbf{e}', \mathbf{x}' \rangle$ respectively. We can now define an invariant coordinate p_1 for $\mathbf{x}$ in the first image as the cross ratio of the four concurrent lines $\langle \mathbf{e}, \mathbf{x}_i \rangle$, $\langle \mathbf{e}, \mathbf{x} \rangle$. Similarly for a coordinate p'_1 for $\mathbf{x}'$ in the second image. The lines used to define p_1 and the lines for p'_1 are projections of the same planes used for P_1. Therefore they have the same cross ratio and we can write

$$P_1 = p_1 = p'_1.$$

For the other components of $\mathbf{P}$ the equality between the two images does not hold since we do not use the same planes for both images. However, there is still an equality between the 3-D and the corresponding 2-D coordinate. The coordinate P_2 was defined above using the four planes

$$\langle \mathbf{O}, \mathbf{O}', \mathbf{X}_1 \rangle,\ \langle \mathbf{O}, \mathbf{X}_2, \mathbf{X}_1 \rangle,\ \langle \mathbf{O}, \mathbf{X}_3, \mathbf{X}_1 \rangle,\ \langle \mathbf{O}, \mathbf{X}, \mathbf{X}_1 \rangle$$

having the common line $\langle \mathbf{O}, \mathbf{X}_1 \rangle$. These planes are projected on the first image as the concurrent lines

$$\langle \mathbf{e}, \mathbf{x}_1 \rangle,\ \langle \mathbf{x}_2, \mathbf{x}_1 \rangle,\ \langle \mathbf{x}_3, \mathbf{x}_1 \rangle,\ \langle \mathbf{x}, \mathbf{x}_1 \rangle.$$

The cross ratio of these lines can be defined as a coordinate p_2 in the first image, which is equal to the 3-D coordinate P_2:

$$P_2 = p_2.$$

Similarly, for the other image one can show

$$P_3 = p_2'.$$

where p_2' is defined in the same way as p_1, interchanging the roles of $\mathbf{O}$, $\mathbf{O}'$.

In summary, we have the 2-D invariant coordinates $(p_1, p_2, 1)^t$ in the first image, $(p_1, p_2', 1)^t$ in the second image, with the relation to the 3-D coordinates given by

$$\mathbf{P} = (P_1, P_2, P_3, 1)^T = (p_1, p_2, p_2', 1)^T. \tag{3.1}$$

As we see from the right hand side in equation (3.1), the vector $\mathbf{P}$ can be calculated by measuring quantities of the two images (given the epipoles). This is true for any choice of camera configurations– the camera planes do not have to coincide as in some previous treatments. The positioning of the camera planes is immaterial since it does not enter the calculation of the cross ratios.

A reconstruction of a shape from uncalibrated cameras can proceed as follows. We assume that we know the position of the two epipoles in the images, $\mathbf{e}, \mathbf{e}'$ and three matching basis point projections, $\mathbf{x}_i$, $\mathbf{x}_i'$. For any additional pair $\mathbf{x}, \mathbf{x}'$ we can now find the 3-D invariant vector $\mathbf{P}$ using the equations above. This vector differs from the the 3-D coordinates $\mathbf{X}$ by some unknown (but fixed) projectivity $\underline{\mathbf{A}}$ as mentioned before. Therefore, reconstruction from uncalibrated cameras can be done up to a 3-D projective transformation of the coordinates $\mathbf{X}$.

3.2 Reconstruction from Matching Conics

We show here a method for reconstructing general scenes from an uncalibrated stereo pair, given matching conics and the epipoles. As we have seen, knowing the epipoles, we need only three more matching points in order to be able to reconstruct any other point in the scene (up to a projectivity). Therefore we have to find three matching points. Two matching points are obvious: the points of contact with the conic of the tangents drawn from the epipole (cf. Figure 3.1). Since tangents and epipoles match, these two points match also. For finding a third point, there are two cases:

i) The fundamental matrix $\underline{\mathbf{F}}$ is known (in addition to the epipoles). In this case we can pass through the epipole a line $\mathbf{l}$ that cuts the conic in two points, $\mathbf{q_1}, \mathbf{q_2}$. Using the fundamental matrix, we find the matching line $\mathbf{l}'$ in the other image (as $\mathbf{l}' = \underline{\mathbf{F}}\mathbf{q_1}$). Finding the intersection of $\mathbf{l}'$ with the conic in the second image, we have the two intersection points $\mathbf{q}_1', \mathbf{q}_2'$, which match $\mathbf{q}_1, \mathbf{q}_2$.

ii) The fundamental matrix is unknown. In this case we need to use another matching conic (or a known matching point).

In the remainder of this section we calculate the intersections of a line with a conic and find the contact points of the tangents drawn from a given point. The treatment is based on that in ([16], p.182).

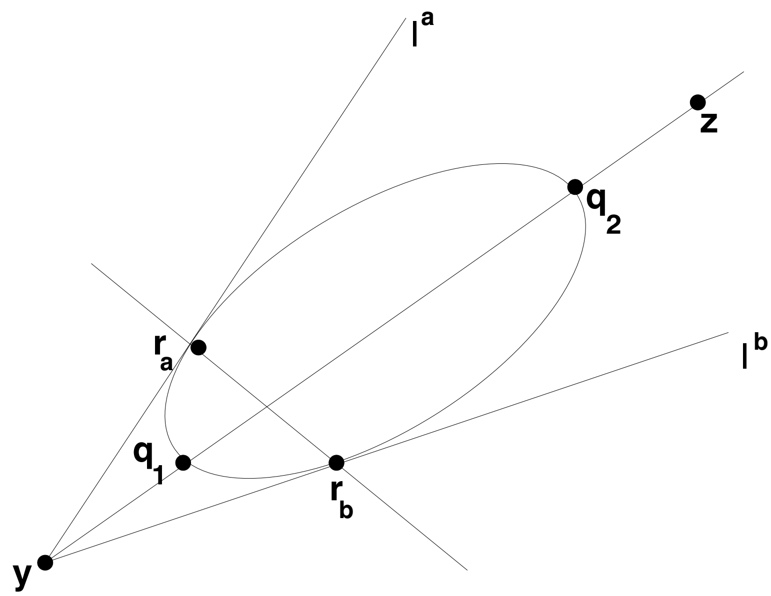

Figure 3.1: Calculating the intersections and contact points of a line with a conic.

We use non-homogeneous triplets of coordinates, $\mathbf{x} = (x^1, x^2, 1)$. We represent the line with the help of two fixed points, $\mathbf{y}, \mathbf{z}$. For our case we will identify $\mathbf{y}$ with the epipole, $\mathbf{y} = \mathbf{e}$. A variable point $\mathbf{x}$ along the line can be represented as the linear combination

$$\mathbf{x} = \lambda \mathbf{y} + \mu \mathbf{z}$$

with the parameters λ, μ satisfying $\lambda + \mu = 1$. To find the intersection of this line with the conic, we substitute the above line in the conic equation $\mathbf{x}^t \underline{\mathbf{C}} \mathbf{x} = 0$. We obtain

$$(\mathbf{y}^t\underline{\mathbf{C}}\mathbf{y})\lambda^2 + 2(\mathbf{y}^t\underline{\mathbf{C}}\mathbf{z})\lambda\mu + (\mathbf{z}^t\underline{\mathbf{C}}\mathbf{z})\mu^2 = 0.$$

We want to solve the above equation along with the condition $\lambda + \mu = 1$. The values we find for λ, μ give the intersection points $\mathbf{q_1}, \mathbf{q_2}$. The above equation is a quadratic equation for λ/μ, with two solutions s_1, s_2. Thus we obtain

$$\frac{\lambda_1}{1-\lambda_1} = s_1.$$

which is easily solved for λ_1. Similarly for s_2.

The case in which the intersections $\mathbf{q_1}, \mathbf{q_2}$ coincide gives us the tangent line. For this case to occur, the discriminant of the quadratic equation above has to vanish:

$$(\mathbf{y}^t\underline{\mathbf{C}}\mathbf{z})^2 - (\mathbf{y}^t\underline{\mathbf{C}}\mathbf{y})(\mathbf{z}^t\underline{\mathbf{C}}\mathbf{z}) = 0$$

We hold $\mathbf{y}$ fixed (the epipole) and use $\mathbf{z}$ as the line variable. The above expression can be decomposed into a product of two factors linear in $\mathbf{z}$, the tangent lines.

We can find the tangents in a simpler and more direct way (without finding intersections). We move the origin to the epipole. All lines through the origin can be expressed as

$$\mathbf{l}(\lambda) = (\lambda, 1, 0).$$

With the free parameter λ here being the slope. We substitute this in the line conic, namely the inverse of $\underline{\mathbf{C}}$:

$$\underline{\mathbf{D}} = \underline{\mathbf{C}}^{-1}.$$

and obtain a quadratic equation in λ:

$$D_{11}\lambda^2 + 2D_{12}\lambda + D_{22}.$$

Using either method we obtain two tangent lines $\mathbf{l}^a, \mathbf{l}^b$. The contact points $\mathbf{r}_a, \mathbf{r}_b$ of the tangents with the conic can be found as the epipolar points of the above lines:

$$\mathbf{r}_a = \mathbf{l}^a\underline{\mathbf{D}},\ \mathbf{r}_b = \mathbf{l}^b\underline{\mathbf{D}}.$$

These contact points match for the two images and can be used for reconstruction.

3.3 Reconstruction from General Curves

In this section we consider the 3-D reconstruction from a stereo pair, given one matching curve and the epipolar geometry. Matching curves is easier than matching points, since there are fewer of them and they are more distinctive.

Instead of matching points, we use derivatives of the curve at any point on the curve. We do not need to have any known matching points. We describe here the extreme case in which no matching points are known and we rely totally on derivatives. However, 'hybrid' methods involving both matching points and derivatives are not hard to devise using the present example and may be even more useful.

Since the epipolar geometry is given, we can draw a line from the epipole to intersect the given curve at point $\mathbf{x}$ in one image. A matching epipolar line intersects the curve in the other image at a matching point $\mathbf{x}'$. Obviously the epipolar line $\underline{\mathbf{F}}\mathbf{x}$ may intersect the curve in more than one point. This may introduce ambiguities which can make it difficult to establish the point correspondences. Instead of intersecting lines we can also consider tangent lines from the epipole to the curve. This reduces the possible ambiguities but it is still possible that the correspondences cannot be established uniquely. However, in most practical cases it will not be too difficult to resolve these ambiguities at least for the tangents by some heuristics.

The basic idea is now to construct a conic that osculates the given curve at $\mathbf{x}$, and a matching conic osculating the matching curve in the other conic at $\mathbf{x}'$. Thus the problem is reduced to the problem solved earlier, namely the reconstruction from one given pair of matching conics plus the epipolar geometry.

The conic in each image has to be determined uniquely. If we rely totally on derivatives, it means that the derivatives of the given curve have to be equal to those of the conic at the point of their contact $\mathbf{x}$, up to fourth order. If a matching point (or line) is known, we need only a second derivative equality. Alternatively, we can use the above method of drawing epipolar lines to find another pair of matching points, say $\mathbf{y}, \mathbf{y}'$. We can then find a conic that osculates the curve at both $\mathbf{x}$ and $\mathbf{y}$, which would require only a second order derivative at each point. Finding a third point $\mathbf{z}$ reduces the situation to the common one of three matching points eliminating the need for derivatives (or conics) altogether.

We will only summarize here the case in which the osculating conic is determined uniquely by the derivatives at $\mathbf{x}$ (or $\mathbf{x}'$). A full treatment of this and the 'hybrid' cases of conic fitting is given in [19, 20].

The first stage is to fit to the data a high order polynomial that represents the given curve f:

$$\begin{aligned} f(x,y) &= a_0 + a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^3 + a_7 x^2 y + \\ &\quad a_8 xy^2 + a_9 y^3 + a_{10} x^4 + a_{11} x^3 y + a_{12} x^2 y^2 + a_{13} xy^3 + a_{14} y^4 = 0. \end{aligned}$$

This quartic implicit polynomial gives a reasonable accurate fit that enables us to calculate derivatives. To simplify the treatment, we next move to a canonical Euclidean system. Namely, we move the origin to the contact point $\mathbf{x}$, and we rotate the axes so that the x axis is tangent to the curve at $\mathbf{x}$. This is easy to accomplish by a Euclidean transformation that sets $a_1 = a_2 = 0$ in the polynomial f above. The remaining coefficients a_i are transformed to new ones in this system, $\bar{a}_i$.

The derivatives in this system are easy to calculate. The zeroth and first one vanish because of the choice of the coordinate system. The higher derivatives are a follows. Setting $a_2 = 1$, denoting $d_n = \frac{1}{n!}\frac{d^n y}{dx^n}(0)$ and dropping the bar from $\bar{a}_i$ we have

$$
\begin{aligned}
d_2 &= -a_3 \\
d_3 &= -a_6 - d_2 a_4 \\
d_4 &= -a_{10} - d_2 a_7 - d_2^2 a_5 - d_3 a_4.
\end{aligned}
$$

In our Euclidean canonical system the osculating conic has only three free coefficients since it passes through the origin and is tangent to x there. We can write it as

$$c_0 x^2 + c_1 y^2 + c_2 xy + y = 0.$$

This conic is determined by the condition that its derivatives at the origin are equal to those of the original curve, i.e. the conic coefficients c_i are determined by d_n. Given d_n it is easy to find the conic:

$$
\begin{aligned}
c_0 &= -d_2 \\
c_1 &= -(d_2 d_4 - d_3^2)/d_2^3 \\
c_2 &= -d_3/d_2.
\end{aligned}
$$

Thus the conic coefficients have been found. We can now return to the original coordinate system and transform the conic to that system.

Chapter 4

Recovering the epipolar geometry from conic correspondences

As we have seen in the previous sections it is possible to solve the problem of projective reconstruction from curve correspondences as soon as the coordinates of the epipoles in the two camera planes are known. We now present a technique for recovering the epipoles and the fundamental matrix $\underline{\mathbf{F}}$ from the correspondence of nondegenerate conics (algebraic curves of second order). It is assumed that these conics are the projections of nondegenerate planar conics in 3D space. A typical setup is shown in Figure 2.1.

4.1 The basic idea

In this section we describe the basic idea in an informal way. Later on we will develop the theory rigorously. Let us assume for the moment that we have a pair of nondegenerate corresponding conics $\underline{\mathbf{C}}, \underline{\mathbf{C}}'$ in the two camera planes and that we know the coordinates of the epipoles $\mathbf{e}, \mathbf{e}'$. From $\mathbf{e}$ we can draw two tangents to the conic $\underline{\mathbf{C}}$. This gives us the two tangent points $\mathbf{x}_1$, $\mathbf{x}_2$ on the conic $\underline{\mathbf{C}}$. These two tangent lines form a degenerate conic of rank two (a cone, cf. Figure A.1) which we denote by $\underline{\mathbf{K}}$. In an analogous manner we get points $\mathbf{x}_1'$, $\mathbf{x}_2'$ and a cone $\underline{\mathbf{K}}'$ in the other image by drawing the tangents from the epipole $\mathbf{e}'$ to the conic $\underline{\mathbf{C}}'$. All this is shown in Figure 4.1.

We say that the cones $\underline{\mathbf{K}}, \underline{\mathbf{K}}'$ are associated with the conics $\underline{\mathbf{C}}, \underline{\mathbf{C}}'$. As already mentioned $\mathbf{x}_1$, $\mathbf{x}_1'$ and $\mathbf{x}_2$, $\mathbf{x}_2'$ are corresponding points, i.e. every pair is the projection from one 3D point to the respective camera planes. The basic idea for recovering the epipoles is to express the the cone $\underline{\mathbf{K}}$ in two different ways. First we use the aforementioned tangent construction for expressing $\underline{\mathbf{K}}$ in terms of the epipole $\mathbf{e}$ and the conic $\underline{\mathbf{C}}$. Second we exploit the restrictions imposed by the epipolar geometry to express $\underline{\mathbf{K}}$ in terms of the fundamental matrix $\underline{\mathbf{F}}$ and a symmetric matrix $\underline{\mathbf{C}}_l'$ derived from the conic $\underline{\mathbf{C}}'$. This will give us an equation of the form $\underline{\mathbf{K}} = \underline{\mathbf{F}}^T \underline{\mathbf{C}}_l' \underline{\mathbf{F}}$ where the left hand side is expressed in terms of $\mathbf{e}, \underline{\mathbf{C}}$ and the right hand side in terms of $\underline{\mathbf{F}}, \underline{\mathbf{C}}'$. All equations will be quadratic

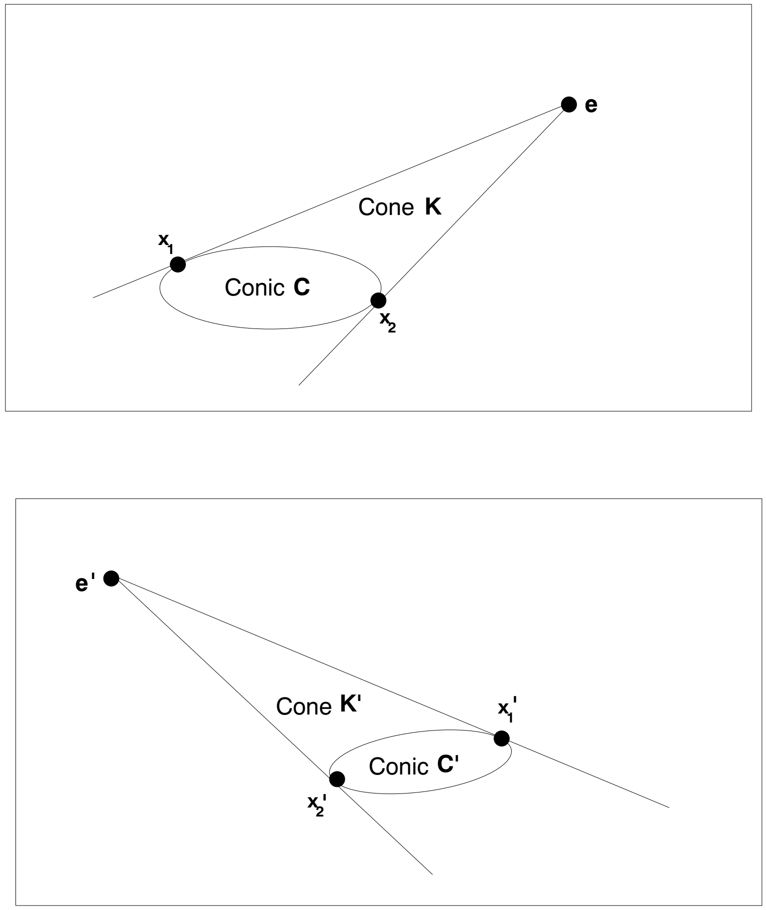

Figure 4.1: Constructing corresponding points and associated cones by drawing tangents from the epipoles $\mathbf{e}, \mathbf{e}'$ to the conics $\underline{\mathbf{C}}, \underline{\mathbf{C}}'$.

in the components of the unknowns $\mathbf{e}, \underline{\mathbf{F}}$. Since we have twelve unknowns (nine from $\underline{\mathbf{F}}$ and three from $\mathbf{e}$) and every corresponding pair of conics yields six equations we should be able to solve for $\mathbf{e}, \underline{\mathbf{F}}$ from the correspondence of two conics.

4.2 Linking conics and associated cones

The purpose of this section is to describe analytically the relation between conics and associated cones. We first collect some useful formulas from projective geometry and explain afterwards how to exploit the restrictions coming from epipolar geometry.

4.2.1 Cones and Line Conics

Given a conic $\underline{\mathbf{C}}$ in $\mathbb{P}^2$ and a point $\mathbf{x}$ outside $\underline{\mathbf{C}}$ we can construct a cone $\underline{\mathbf{K}}$ by drawing the two lines through $\mathbf{x}$ which are tangent to $\underline{\mathbf{C}}$. As shown in ([16], p. 183) the matrix $\underline{\mathbf{K}}$ with components K_{ij} is given by (note that we use the Einstein summation convention):

$$K_{ij} = (C_{kl}x^k x^l)C_{ij} - (C_{ki}x^k)(C_{lj}x^l). \tag{4.1}$$

This can also be written in matrix/vector notation (keep in mind that the matrix $\underline{\mathbf{C}}$ is symmetric, $\underline{\mathbf{C}}^T = \underline{\mathbf{C}}$):

$$\underline{\mathbf{K}} = \mathbf{x}^T\underline{\mathbf{C}}\mathbf{x}\underline{\mathbf{C}} - \underline{\mathbf{C}}\mathbf{x}\mathbf{x}^T\underline{\mathbf{C}}. \tag{4.2}$$

In order to take advantage of the restrictions imposed by the epipolar geometry we need the so called line conic. For a nondenerate conic $\mathcal{C}$ we can draw for every point $\mathbf{x} \in \mathcal{C}$ on the conic exactly one line through $\mathbf{x}$ which is tangent to $\mathcal{C}$. The set of all lines which are tangent to a conic $\mathcal{C}$ is also a conic. This means that for every nondegenerate conic $\mathcal{C}$ it is possible to construct a nondegenerate symmetric matrix $\underline{\mathbf{C}}_l$ so that the set of solutions of the quadratic equation $\mathbf{l}^T\underline{\mathbf{C}}_l\mathbf{l}$ coincides with the set of line coordinates describing lines which are tangent to the conic $\mathcal{C}$. $\underline{\mathbf{C}}_l$ is called the line conic associated to the conic $\underline{\mathbf{C}}$. In order to construct the line conic $\underline{\mathbf{C}}_l$ one needs the cofactors C^{ij} of the matrix $\underline{\mathbf{C}}$. For a given matrix $\underline{\mathbf{C}}$ and integers $r, s \in \mathbb{N},\ r, s \leq \dim C$ we construct the reduced matrix $\underline{\mathbf{C}}_{(r,s)}$ by removing row r and column s from $\underline{\mathbf{C}}$. The cofactor C^{ij} is then defined by

$$C^{ij} := (-1)^{i+j} \det \underline{\mathbf{C}}_{(i,j)}. \tag{4.3}$$

As shown in ([16], p. 173) the components $C_{l_{ij}}$ of the line conic $\underline{\mathbf{C}}_l$ are given by

$$C_{l_{ij}} = C^{ij}. \tag{4.4}$$

4.2.2 Combining conic correspondences with the epipolar geometry

Now we are ready to prove our central result which allows the recovery of the epipolar geometry from correspondences of nondegenerate conics. We assume that we have established the correspondence of two nondegenerate conics $\underline{\mathbf{C}}$ and $\underline{\mathbf{C}}'$ in the two images. We denote the epipoles by $\mathbf{e}$ and $\mathbf{e}'$. As described in section 4.2.1 we construct two cones $\underline{\mathbf{K}}$ and $\underline{\mathbf{K}}'$ by drawing the tangents to $\underline{\mathbf{C}}, \underline{\mathbf{C}}'$ which pass through the epipoles $\mathbf{e}, \mathbf{e}'$. Let $\underline{\mathbf{F}}$ be the fundamental matrix of the stereo setup and let $\underline{\mathbf{C}}_l, \underline{\mathbf{C}}_l'$ be the line conics corresponding to the conics $\underline{\mathbf{C}}, \underline{\mathbf{C}}'$.

Theorem 4.1 *Let* $\underline{\mathbf{C}}$ *and* $\underline{\mathbf{C}}'$ *be a pair of corresponding nondegenerate conics. We denote the epipoles by* $\mathbf{e}$ *and* $\mathbf{e}'$*. Let* $\underline{\mathbf{K}}$ *and* $\underline{\mathbf{K}}'$ *be the two cones constructed by drawing the tangents to* $\underline{\mathbf{C}}, \underline{\mathbf{C}}'$ *which pass through the epipoles* $\mathbf{e}, \mathbf{e}'$*. Let* $\underline{\mathbf{F}}$ *be the fundamental matrix of the stereo rig and let* $\underline{\mathbf{C}}_l'$ *be the line conic corresponding to the conic* $\underline{\mathbf{C}}'$*. The cone* $\underline{\mathbf{K}}$ *determined by the epipole* $\mathbf{e}$ *and the conic* $\underline{\mathbf{C}}$ *can be described by:*

1. *using the conic* $\underline{\mathbf{C}}$ *and the epipole* $\mathbf{e}$*:*

$$\underline{\mathbf{K}} = \mathbf{e}^T \underline{\mathbf{C}} \mathbf{e} \underline{\mathbf{C}} - \underline{\mathbf{C}} \mathbf{e} \mathbf{e}^T \underline{\mathbf{C}} \tag{4.5}$$

2. *using the line conic* $\underline{\mathbf{C}}_l'$ *and the fundamental matrix* $\underline{\mathbf{F}}$*:*

$$\underline{\mathbf{K}} = \underline{\mathbf{F}}^T \underline{\mathbf{C}}_l' \underline{\mathbf{F}}. \tag{4.6}$$

By interchanging in equations (4.5), (4.6) primed with unprimed quantities and $\underline{\mathbf{F}}$ *with* $\underline{\mathbf{F}}^T$ *it is also possible to express the cone* $\underline{\mathbf{K}}'$ *in two different ways.*

Proof: Equation (4.5) is a trivial reformulation of equation (4.2). Since $\underline{\mathbf{F}}$ has rank two the matrix $\underline{\mathbf{F}}^T \underline{\mathbf{C}}_l' \underline{\mathbf{F}}$ has also rank two (the line conic $\underline{\mathbf{C}}_l'$ has rank three) and describes therefore a cone. In order to prove the equivalence of (4.5) and (4.6) it is enough to show that the two cones have the same apex $\mathbf{a}$ and to identify two further points $\mathbf{x}_1$, $\mathbf{x}_2$ which lie on both cones so that the three points $\mathbf{a}$, $\mathbf{x}_1$, $\mathbf{x}_2$ are not collinear. The apex $\mathbf{a}$ of a cone $\underline{\mathbf{K}}$ is characterized by the condition $\underline{\mathbf{K}}\mathbf{a} = \mathbf{0}$. This makes it easy to show that the epipole $\mathbf{e}$ is the apex for the two cones in (4.5), (4.6) (note that $\underline{\mathbf{F}}\mathbf{e} = \mathbf{0}$, cf. equation (2.5)). Let $\mathbf{x}_1$, $\mathbf{x}_2$ be the two points on the conic $\underline{\mathbf{C}}$ which are on the tangent from $\mathbf{e}$ to $\underline{\mathbf{C}}$. Trivially these points are on the cone $\underline{\mathbf{K}}$ described in equation (4.5). We have to prove that

$$\mathbf{x}_i^T \underline{\mathbf{F}}^T \underline{\mathbf{C}}_l' \underline{\mathbf{F}} \mathbf{x}_i = 0 \; \forall i = 1, 2. \tag{4.7}$$

We set $\mathbf{l}_i' := \underline{\mathbf{F}}\mathbf{x}_i$. The fundamental matrix $\underline{\mathbf{F}}$ maps points from the unprimed image to lines in the primed image which pass through the epipole so that the matching point lies on this line. The points corresponding to $\mathbf{x}_1$, $\mathbf{x}_2$ are on the conic $\underline{\mathbf{C}}'$. Therefore the lines $\mathbf{l}_1'$, $\mathbf{l}_2'$ are on the line conic $\underline{\mathbf{C}}_l'$ which means $\mathbf{l}_i'^T \underline{\mathbf{C}}_l' \mathbf{l}_i' = 0 \; \forall i = 1, 2$. □

4.3 Computing the fundamental matrix and the epipoles

By using the results from Theorem 4.1 it is possible to determine the fundamental matrix $\underline{\mathbf{F}}$ and the epipoles $\mathbf{e}$, $\mathbf{e}'$ from conic correspondences. Every pair of matching conics $\underline{\mathbf{C}}, \underline{\mathbf{C}}'$ yields according to Theorem 4.1 the following matrix equation:

$$\underline{\mathbf{F}}^T \underline{\mathbf{C}}_l^{'} \underline{\mathbf{F}} = \mathbf{e}^T \underline{\mathbf{C}} \mathbf{e} \underline{\mathbf{C}} - \underline{\mathbf{C}} \mathbf{e} \mathbf{e}^T \underline{\mathbf{C}}. \tag{4.8}$$

Since all matrices are symmetric this yields six equations. The rank of the matrices is two, therefore the equations cannot be independent. The correspondence of a pair of conics gives twelve equations which we can eventually solve for the twelve unknowns $\underline{\mathbf{F}}, \mathbf{e}$. Since $\underline{\mathbf{F}}$ has rank two the 'unknowns' are also dependent.

If we have $\underline{\mathbf{F}}, \mathbf{e}$ then the coordinates of the other epipole $\mathbf{e}^{'}$ can be determined as follows. We draw the tangents from the epipole $\mathbf{e}$ to the conic $\underline{\mathbf{C}}$. The two points where the tangents touch the conic are denoted by $\mathbf{x}_1,\ \mathbf{x}_2$. The intersection point of the two epipolar lines $\underline{\mathbf{F}}\mathbf{x}_1,\ \underline{\mathbf{F}}\mathbf{x}_2$ in the primed image is the epipole $\mathbf{e}^{'}$.

In order to solve the equations (4.8) for $\underline{\mathbf{F}}, \mathbf{e}$ it is advantageous to make use of the standard forms for pairs of conics developed in Theorem A.3. It is possible to obtain a solution numerically by a constrained minimization algorithm (note that $\underline{\mathbf{F}}, \mathbf{e}$ are not independent, cf. equation (2.5)). First simulations have indicated that a straightforward implementation is not stable and requires a good initial guess. Furthermore it is at present an open question how to solve the equations (4.8) analytically and how to characterize possible ambiguities.

Chapter 5

Summary and Conclusion

We have investigated in this paper methods for 3-D reconstruction from uncalibrated stereo views using correspondences of curves. This extends previous work which was based on correspondences of points or lines. First we have introduced a coordinate system which is invariant to the (unknown) camera parameters. This construction may be useful in other contexts as well. Based on this coordiante system we have shown how to achieve a 3-D reconstruction from an uncalibrated stereo pair using correspondences of conics. This was extended afterwards to the case of general curves. A prerequisite for these methods was the knowledge of the epipolar geometry. We have presented a method for determining the fundamental matrix and the coordinates of the epipoles from the correspondences of a pair of conics. We have not discussed in this paper implementation details or questions concerning the robustness of the solutions. This are important points for further work.

Appendix A

Geometric equivalence and normal forms of conics

Two conics $\mathcal{C}_1, \mathcal{C}_2$ with describing matrices $\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2$ are called geometrically equivalent if a nonsingular matrix $\underline{\mathbf{A}} \in GL(\mathbb{P}^2)$ exists with $\underline{\mathbf{C}}_1 = \underline{\mathbf{A}}^T \underline{\mathbf{C}}_2 \underline{\mathbf{A}}$. The significance of this concept lies in the fact that geometrically equivalent conics only differ with respect to the coordinate systems which are used for their analytical description. How is it possible to decide whether two conics are geometrically equivalent or not? The answer is provided by the following classification theorem (cf. [9]).

Theorem A.1 *Two conics $\mathcal{C}_1, \mathcal{C}_2$ with describing matrices $\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2$ are geometrically equivalent if and only if rank $\underline{\mathbf{C}}_1$ = rank $\underline{\mathbf{C}}_2$. Every conic $\mathcal{C}$ in $\mathbb{P}^2$ is geometrically equivalent to one of following conics which are called normal forms:*

rank $\underline{\mathbf{C}}$	*Equation*	*Description*
0	$0 = 0$	$\mathbb{P}^2$
1	$x_1^2 = 0$	*(double) line*
2	$x_1^2 + x_2^2 = 0$	*pair of lines*
3	$x_1^2 + x_2^2 - x_3^2 = 0$	*circle*

For the applications in this paper for the reconstruction from conic correspondences in uncalibrated views only the cases rank $\underline{\mathbf{C}} = 2$ and rank $\underline{\mathbf{C}} = 3$ will be relevant. Figure A.1 shows the two normal forms and some transformed conics.

In the case rank $\underline{\mathbf{C}} = 2$ we will also use the term cone instaed of conic. Apart from normal forms for single conics we will also need normal forms for pairs of nondegenerate conics. For tackling these probblems we have to use results concerning invariants of pairs of conics. An invariant $I(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2)$ of two conics $\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2$ is a function of the conic coefficients which remains unchangend under transformations of the conic pair, i.e. for every nonsingular matrix $\underline{\mathbf{A}} \in GL(\mathbb{P}^2)$ we have

$$I(\underline{\mathbf{A}}^T \underline{\mathbf{C}}_1 \underline{\mathbf{A}}, \underline{\mathbf{A}}^T \underline{\mathbf{C}}_2 \underline{\mathbf{A}}) = I(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2).$$

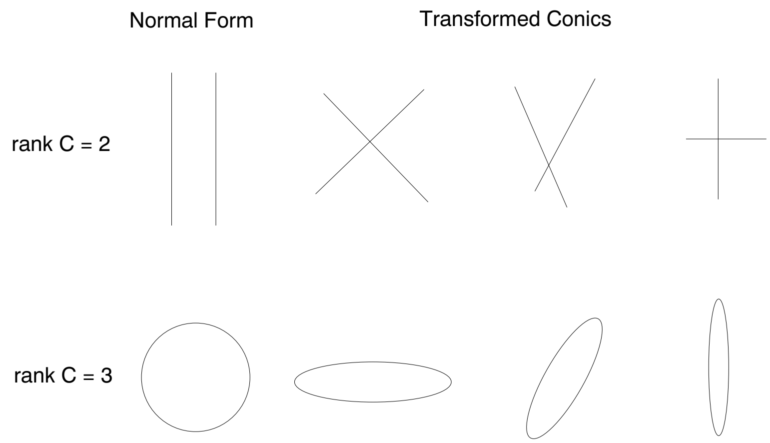

Figure A.1: Normal forms and transformed conics for the cases rank $\underline{\mathbf{C}} = 2$ and rank $\underline{\mathbf{C}} = 3$.

It is well known (cf. [16, 14]) that a pair of nondegenerate conics $\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2$ in $\mathbb{P}^2$ has the following two basic invariants:

$$\begin{aligned} I_1(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2) &= \left(\frac{\det \underline{\mathbf{C}}_1}{\det \underline{\mathbf{C}}_2}\right)^{\frac{1}{3}} \mathrm{Trace}(\underline{\mathbf{C}}_1^{-1}\underline{\mathbf{C}}_2) \\ I_2(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2) &= \left(\frac{\det \underline{\mathbf{C}}_2}{\det \underline{\mathbf{C}}_1}\right)^{\frac{1}{3}} \mathrm{Trace}(\underline{\mathbf{C}}_1\underline{\mathbf{C}}_2^{-1}). \end{aligned} \tag{A.1}$$

An important point for our purposes is that the values of the invariants I_1, I_2 from equation (A.1) characterize a pair of nondegenerate conics $\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2$ uniquely up to projective transformation. That means precisely the following.

Theorem A.2 *Let two pairs of nondegenerate conics* $(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2)$ *and* $(\underline{\mathbf{D}}_1, \underline{\mathbf{D}}_2)$ *in* $\mathbb{P}^2$ *be given. If the values of the invariants* I_1, I_2 *coincide for the two pairs of conics, i.e.* $I_1(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2) = I_1(\underline{\mathbf{D}}_1, \underline{\mathbf{D}}_2)$ *and* $I_2(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2) = I_2(\underline{\mathbf{D}}_1, \underline{\mathbf{D}}_2)$ *then there exists a nonsingular matrix* $\underline{\mathbf{A}} \in GL(\mathbb{P}^2)$ *with* $(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2) = (\underline{\mathbf{A}}^T\underline{\mathbf{D}}_1\underline{\mathbf{A}}, \underline{\mathbf{A}}^T\underline{\mathbf{D}}_2\underline{\mathbf{A}})$.

Theorem A.2 enables us to prove some useful results concerning normal forms of pairs of conics. What we are interested in is to find for a pair of nondegenerate conics $(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2)$ a nonsingular matrix $\underline{\mathbf{A}} \in GL(\mathbb{P}^2)$ so that the transformed pair $(\underline{\mathbf{A}}^T\underline{\mathbf{C}}_1\underline{\mathbf{A}}, \underline{\mathbf{A}}^T\underline{\mathbf{C}}_2\underline{\mathbf{A}})$ has a particularly simple form. We will show that it is possible to find matrices $\underline{\mathbf{A}} \in GL(\mathbb{P}^2)$ so that the transformed conics are diagonal. These diagonal forms are not unique. In order to give a compact description of the arising ambiguities we need some additional terminology. We denote by I_1, I_2 the values of the invariants of equation (A.1) for the two nondegenerate conics $(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2)$. We define a polynomial $f(x)$ (x a complex variable) as follows:

$$f(x) := x^3 + I_1 x^2 + I_2 x + 1. \tag{A.2}$$

Let $R(f)$ denote a root of the polynomial f, i.e. $f(R(f)) = 0$. For a root $R(f)$ the quadratic polynomial $g_{R(f)}(x)$ is defined by:

$$g_{R(f)}(x) := x^2 - x(I_1 - R(f)) + I_2 + I_1 R(f) + R(f)^2. \tag{A.3}$$

Using the polynomials from equations (A.2), (A.3) it is possible to give a complete description of all possible diagonal forms of pairs of nondegenerate conics.

Theorem A.3 *Let $(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2)$ be a pair of nondegenerate conics and denote by I_1, I_2 the values of the invariants of equation (A.1) for these conics. Let $(\underline{\mathbf{D}}_1, \underline{\mathbf{D}}_2)$ be a pair of diagonal matrices of the following form:*

$$\begin{aligned} \underline{\mathbf{D}}_1 &= diag\,(1, 1, -1) \\ \underline{\mathbf{D}}_2 &= diag\,(a, b, c) \\ with \quad a &= \frac{-1}{R(f) R(g_{R(f)})},\ b = R(g_{R(f)}),\ c = R(f). \end{aligned} \tag{A.4}$$

Then there exists a nonsingular matrix $\underline{\mathbf{A}} \in GL(I\!P^2)$ with $(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2) = (\underline{\mathbf{A}}^T \underline{\mathbf{D}}_1 \underline{\mathbf{A}}, \underline{\mathbf{A}}^T \underline{\mathbf{D}}_2 \underline{\mathbf{A}})$.

Proof: The parameters a, b, c are constructed in such a manner that they describe the set of all solutions of the following system of equations (verification by direct calculation or using MAPLE):

$$\begin{aligned} I_1 &= a + b - c \\ I_2 &= \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \\ -1 &= abc \end{aligned} \tag{A.5}$$

A direct calculation gives for the values of the invariants (equation (A.1)) for the pair of conics $(\underline{\mathbf{D}}_1, \underline{\mathbf{D}}_2)$:

$$I_1(\underline{\mathbf{D}}_1, \underline{\mathbf{D}}_2) = a + b - c$$

$$I_2(\underline{\mathbf{D}}_1, \underline{\mathbf{D}}_2) = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

This yields together with equations (A.5) $I_1(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2) = I_1(\underline{\mathbf{D}}_1, \underline{\mathbf{D}}_2)$ and $I_2(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2) = I_2(\underline{\mathbf{D}}_1, \underline{\mathbf{D}}_2)$. Since the values of the two fundamental invariants coincide for $(\underline{\mathbf{C}}_1, \underline{\mathbf{C}}_2)$ and $(\underline{\mathbf{D}}_1, \underline{\mathbf{D}}_2)$ the assertion follows from Theorem A.2. □

The most useful interpretation of Theorem A.3 for our purposes is the following. Given a pair of conics in the camera plane it is always possible to introduce new image coordinates so that one conic becomes a circle around the origin and the other conic an ellipse around the origin. Some of these normal forms are shown in Figure A.2. We note that it is essential here that we work in the complex projective space $I\!P^2$ since we have to determine the roots of the polynomials $f, g_{R(f)}$ (equations (A.2), (A.3)) which are in general complex.

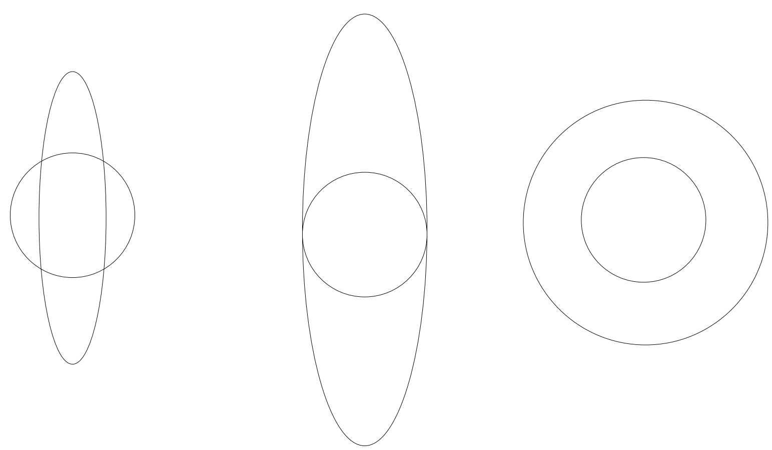

Figure A.2: Examples of normal forms for pairs of conics.

Appendix B

List of Symbols

$\mathbb{P}^n$	$n-$dimensional complex projective space
$\mathbf{x}$	homogeneous coordinates of a point in $\mathbb{P}^2$
$\mathbf{X}$	homogeneous coordinates of a point in $\mathbb{P}^3$
$\mathbf{l}$	homogeneous coordinates of a line
$\langle \mathbf{X}_1, \mathbf{X}_2 \rangle$	line spanned by the two points $\mathbf{X}_1, \mathbf{X}_2$
$\langle \mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3 \rangle$	plane spanned by the three points $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$
$\mathbf{0}$	vector $(0,0,0)^T$
$M(\mathbb{P}^2)$	set of all projective 3×3 matrices
$GL(\mathbb{P}^2)$	set of all nonsingular projective 3×3 matrices
$\mathcal{C}$	conic in $\mathbb{P}^2$
$\mathcal{C}_l$	line conic associated to the conic $\mathcal{C}$
C	symmetric matrix in $M(\mathbb{P}^2)$ describing a conic
C_{ij}	component of the matrix C
C^{ij}	cofactor of the matrix C
$C_{(r,s)}$	reduced matrix
C_l	symmetric matrix in $M(\mathbb{P}^2)$ describing a line conic
$R(f)$	root of a polynomial f
$\underline{\mathbf{F}}$	fundamental matrix of the stereo rig
$\mathbf{e},\ \mathbf{e}'$	coordinates of the epipoles in the two images
$P,\ P'$	projection matrices from 3D to the two retinas

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